Classical Mechanics
Phy 235, Review, Exam 2.

Frank L. H. Wolfs
Department of Physics and Astronomy
University of Rochester

Another interesting fact: how to earn a KLM house.

Exam # 2

- Exam # 2 will take place on Thursday October 23 between 8 am and 9.20 am in B&L 109.
- The exam will cover the material discussed in Chapters 5 – 7.
- The exam will have 4 questions:
  - Three questions will be analytical questions.
  - One question will be a conceptual questions.
- You will be provided with an equation sheet (the same one used for Exam # 1).

Time management

- Work no more than 10 – 15 minutes on each problem.
- Even if not finished, move on to the next problem.
- This will leave 15 minutes at the end to finish your problems and/or make correction.
- We can only give credit for what you write (not what you think).
- We can only give credit for what we can read (write neatly).

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Warning.

- I cannot cover everything I discussed in lectures 7 – 11 in this review.
- If I skip over certain topics, it does not mean you should not understand that material.
- Your TAs will not see the exam until you see it.
- **NOTE**: answer the correct question in the correct booklet.

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How to get help to prepare for Exam # 2.

- Both recitations on Tuesday will be Q&A sessions focused on Exam # 2.
- Everyone can attend one or both recitations on Tuesday.
- There will be regular office hours on Wednesday.
- There will be no office hours on Thursday and there will be no recitation on Thursday.

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Overview

- Chapter 5: Gravitation.
- Chapter 6: Calculus of Variations.
- Chapter 7: Lagrangian and Hamiltonian Dynamics.
  - Note: Sections 7.12 and 7.13 are not included.

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Chapter 5.

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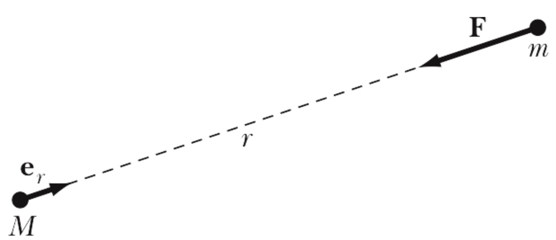
The gravitational force between point particles.



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Gravitational Potential

- Gravitational potential:

$$\vec{g} = -\vec{\nabla}\Phi$$

- Gravitational potential due to a point mass:

$$\Phi = -G\frac{M}{r}$$

- Gravitational potential due to a continuous mass distribution:

$$\Phi = -G\int_V \frac{\rho(\vec{r}\,')}{r'}dv'$$

- Note: the gravitational potential is a scalar.

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Poisson's Equation.

- Gravitational flux due to a point mass *m*:

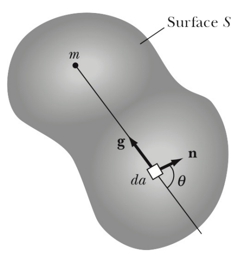
$$\Phi_{flux} = -4\pi G m$$

- When we have a mass distribution inside *S*:

$$\Phi_{flux} = -4\pi G\int_V \rho dv$$

- This relation can be used to show that the gravitational potential satisfies the following equation:

$$\vec{\nabla}^2\Phi_{flux} = 4\pi G\rho$$



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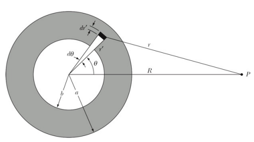
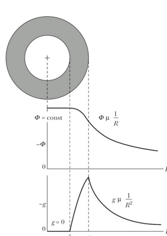
Shell theorem.





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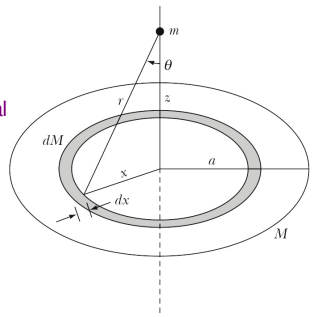
Use symmetry to calculate the net force.

- Calculate the force directly.
- Calculate the potential energy and then determine the force.



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Chapter 6.

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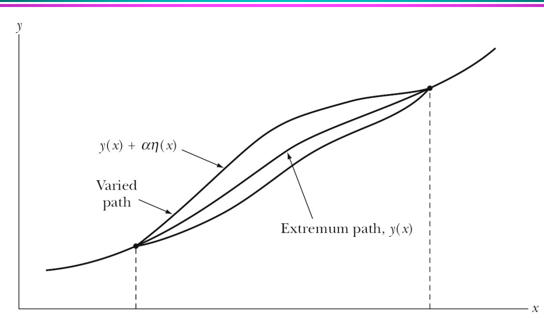
Calculus of Variations: find the path $y(x)$ that minimizes a path integral.



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First version of Euler's equation.

- Goal: minimize the path integral of a function f:

$$J = \int_{x_1}^{x_2} f(y(\alpha, x), y'(\alpha, x); x)dx$$

- Note: x does **NOT** have to be a position; it can for example be time.
- The function f that minimizes J must satisfy the following requirement:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

- This is the first version of Euler's equation.

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Euler's equation with more than one dependent variable.

- Consider the function f which depends on several dependent variables y_1, y_2, y_3, etc.
- In this case, to minimize the path integral of f, the dependent variables must satisfy the following condition:

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right) = 0$$

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Second version of Euler's equation.

- A second version of Euler's equation is useful when f does not explicitly depend on x.
- The second version of Euler's equation is:

$$\frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0$$

- When f does not explicitly depend on x, this equation becomes:

$$f - y'\frac{\partial f}{\partial y'} = \text{constant}$$

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Euler's equations with constraints.

- Consider path constraints: $g\{y, z; x\} = 0$.
- Euler's equations are now:

$$\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right) + \lambda(x)\left(\frac{\partial g}{\partial y}\right) = 0$$

$$\left(\frac{\partial f}{\partial z} - \frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right)\right) + \lambda(x)\left(\frac{\partial g}{\partial z}\right) = 0$$

- The function $\lambda(x)$ is the **Lagrange undetermined multiplier**.

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Problem 6.16

- What curve on the surface $z = x^{3/2}$ joining the points $(x, y, z) = (0, 0, 0)$ and $(1, 1, 1)$ has the shortest arc length?

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Chapter 7.

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Hamilton's Principle

" *Of all the possible paths along which a dynamical system may move from one point to another within a specified time interval (consistent with any constraints), the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energies.* "

$$\delta \int_{t_1}^{t_2} (T - U)dt = 0$$

The quantity T - U is called the **Lagrangian** $\boldsymbol{L}$.

Note: time is the independent variable.

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Hamilton's Principle – Part 2.

- Hamilton's principle: " *Of all the possible paths along which a dynamical system may move from one point to another in configuration space within a specified time interval (consistent with any constraints), the actual path followed is that which minimizes the time integral of the Lagrangian function for the system.* "

$$\delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)dt = 0$$

- **Note**: the **generalized coordinates** q are coordinates that completely specify the state of the system. They do **not** need to be coordinates of a coordinate system.

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The Lagrange Equation(s) of Motion.

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial x_i'}\right) = 0$$

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Lagrange's Equations with Undetermined Multipliers.

- Assume constraints can be expressed in differential form:

$$\sum_{j=1}^{s} \frac{\partial f_k}{\partial q_j} dq_j = 0$$

- Constraints can be incorporated into the Lagrange equations:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \sum_{k=1}^{m} \lambda_k(t) \frac{\partial f_k}{\partial q_j} = 0$$

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What is the use of the undetermined multipliers?

- The forces of constrained can be determined from the equations of constraint and the Lagrange multipliers:

$$Q_j = \sum_{k=1}^{m} \lambda_k(t) \frac{\partial f_k}{\partial q_j}$$

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Generalized coordinates: position, velocity, and momentum.

- So far, we have expressed the Lagrangian in terms of (generalized) position and (generalized) velocities:

$$L = T - U = \frac{1}{2} m \sum_{i=1}^{3} \dot{x}_i^2 - U(x_i)$$

- An alternative is to express the Lagrangian in terms of (generalized) position and (generalized) momenta. For example:

$$p_r = \frac{\partial L}{\partial \dot{r}} = \frac{\partial T}{\partial \dot{r}} = m\dot{r}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

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Conservation Laws – Part I.

- Conservation of energy:
  - Lagrangian does not depend on time explicitly.
  - If L does not depend explicitly on time, it can be shown that

$$L - \sum_j \left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j} \right) = \text{constant} = -H$$

  - The constant H is called the Hamiltonian of the system:

$$H = \sum_j \left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j} \right) - L$$

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Conservation Laws – Part II.

- Conservation of linear momentum:
  - Lagrangian should not be effected by a translation of space.
- Conservation of angular momentum:
  - Lagrangian should not be effected by a rotation of space.

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Canonical Equations of Motion

Lagrange equations of motion in terms of generalized momentum:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \dot{p}_i = \frac{\partial L}{\partial q_i}$$

The Hamiltonian H can be written in terms of the generalized momenta as

$$H = \sum_j \left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j} \right) - L = \sum_j \dot{q}_j p_j - L$$

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Hamilton's Equations of Motion

- For each coordinate: **two** equations of motion.
  - For each coordinate there is only one Lagrange equation of motion.
- Equations of motion are **first order differential equations**.
  - The Lagrange equations of motion are second order differential equations.

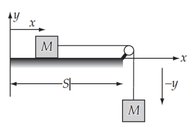
$$\frac{\partial H}{\partial p_j} - \dot{q}_j = 0$$

$$\frac{\partial H}{\partial q_j} + \dot{p}_j = 0$$

$$\frac{\partial H}{\partial t} + \frac{\partial L}{\partial t} = 0$$

Problem 7.10.

- Two blocks of mass *M* are connected by a uniform string of length *l*. One block is placed on a smooth horizontal surface and the other blocks hangs over the side, the string passing over a frictionless pulley. Describe the motion of the system (a) when the mass of the string is negligible and (b) when the string has a mass *m*.



ENOUGH FOR TODAY?