Classical Mechanics
Phy 235, Lecture 20.

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And look at this one ….
My favorite plane and no smog in Beijing.

PH-BFT at PEK (Beijing)

Yesterday was Dutch-American Heritage Day

Dutch-American Heritage Day marks the longstanding history and shared bonds between the Netherlands and the United States.

In 1776, the Netherlands became the first country to formally recognize the United States of America. On that day, the governor of Sint Eustasius ordered the island's cannons fired in response to the 13-gun salute from the Andrew Doria as it sailed into the harbor of the Dutch island. Only four months before, the United States declared its independence from Great Britain. This simple act is recorded as the first salute to the American flag by a foreign nation.

The ties between the Netherlands and New York extend further back to 1609 when Henry Hudson sailed the Dutch ship, the Half Moon, into what is now known as New York Harbor. Hudson's voyage led to the founding of New Netherlands and the trading post New Amsterdam.

From: http://nlintheusa.com/nlus/

Course Announcements.

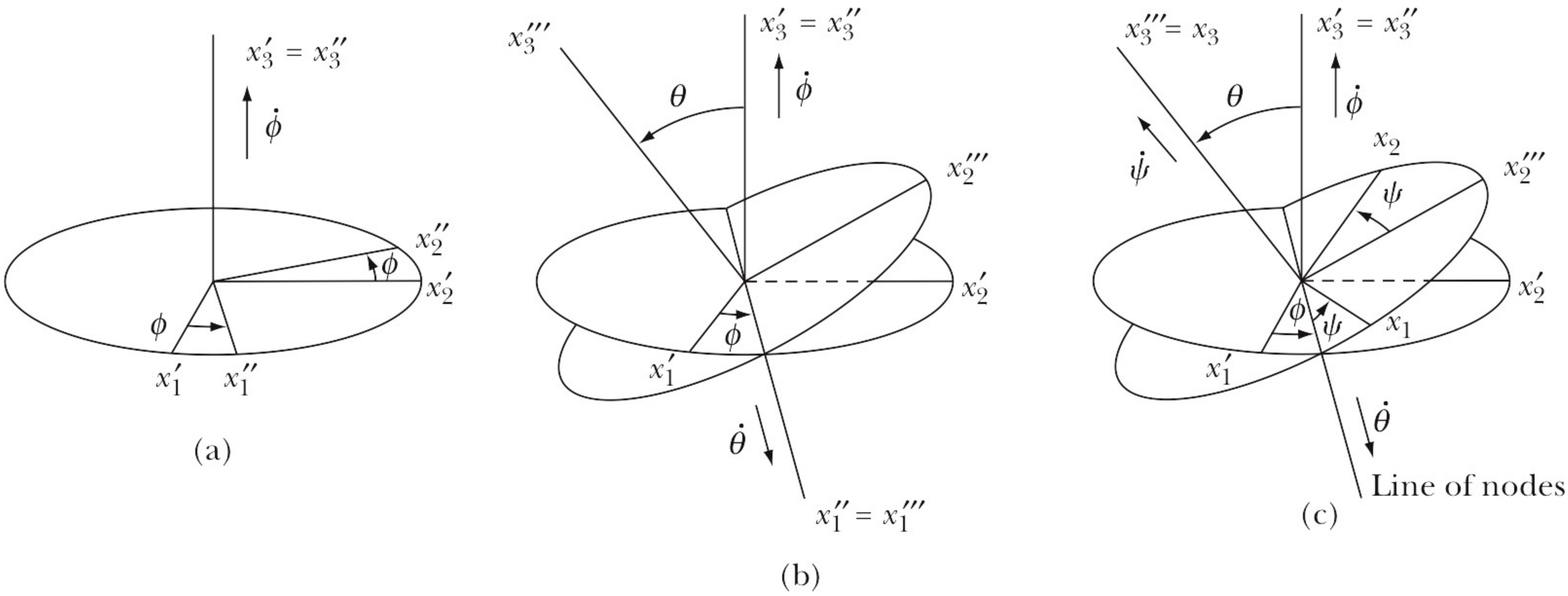
- Exam 3:
  - Tuesday November 18, 8.00 am – 9.20 am (B&L 109).
  - Extra office hours today between 3 pm and 7 pm in the POA.

- Homework set # 10 (the last homework assignment) will be due on Friday December 5.

Course Announcements.

- The term paper for this course is due on Wednesday November 26. Please remember that the first draft of your paper must be discussed with the fellows in the writing center. I will receive a confirmation from your meeting.
- NOTE: the 10 pages for the term paper is a guideline. It will be difficult to discuss your topic with appropriate details in less than 10 pages. A few additional pages is not a problem. But I do not want a 100-page paper either!
- Make sure you refer to all the instructions on the PHY 235 webpages.

Euler Angles.
A quick review.



Note: *x*' is fixed reference frame, *x* is body reference frame.

Euler Angles.
Transformation Matrix.

Rotation around x_3''' axis. ↓

Rotation around x_1'' axis. ↓

Rotation around x_3' axis. ↓

$$\lambda = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

Euler Angles.
Angular Velocity.

$$\bar{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} \dot{\phi}_1 + \dot{\theta}_1 + \dot{\psi}_1 \\ \dot{\phi}_2 + \dot{\theta}_2 + \dot{\psi}_2 \\ \dot{\phi}_3 + \dot{\theta}_3 + \dot{\psi}_3 \end{pmatrix} = \begin{pmatrix} \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \dot{\phi}\cos\theta + \dot{\psi} \end{pmatrix}$$

Lagrange's equations for the three Euler angles.

- We can obtain a Lagrange's equation for each Euler angle:
  - ϕ :

$$\frac{d}{dt}\left\{I_1\omega_1 \sin\theta \sin\psi + I_2\omega_2 \sin\theta \cos\psi + I_3\omega_3 \cos\theta\right\} = 0$$

  - θ :

$$\dot{\phi}\left(\left\{I_1\omega_1 \sin\psi + I_2\omega_2 \cos\psi\right\}\cos\theta - I_3\omega_3 \sin\theta\right) -$$

$$\frac{d}{dt}\left\{I_1\omega_1 \cos\psi - I_2\omega_2 \sin\psi\right\} = 0$$

  - ψ :

$$\left(I_1 - I_2\right)\omega_1\omega_2 - I_3\dot{\omega}_3 = 0 \longleftarrow$$

Only equation that contains just angular velocities.

Lagrange's equations for the three Euler angles.

- Since our choice of coordinate axes was arbitrary, we can find the following relations for the three components of the angular velocity:

$$\left(I_1 - I_2\right)\omega_1\omega_2 - I_3\dot{\omega}_3 = 0$$

$$\left(I_2 - I_3\right)\omega_2\omega_3 - I_1\dot{\omega}_1 = 0$$

$$\left(I_3 - I_1\right)\omega_3\omega_1 - I_2\dot{\omega}_2 = 0$$

Example: symmetric top.

- Two different principle moments: $I_1 = I_2$ and I_3.
- One of the Euler equations tells us:

$$I_3\dot{\omega}_3 = 0$$

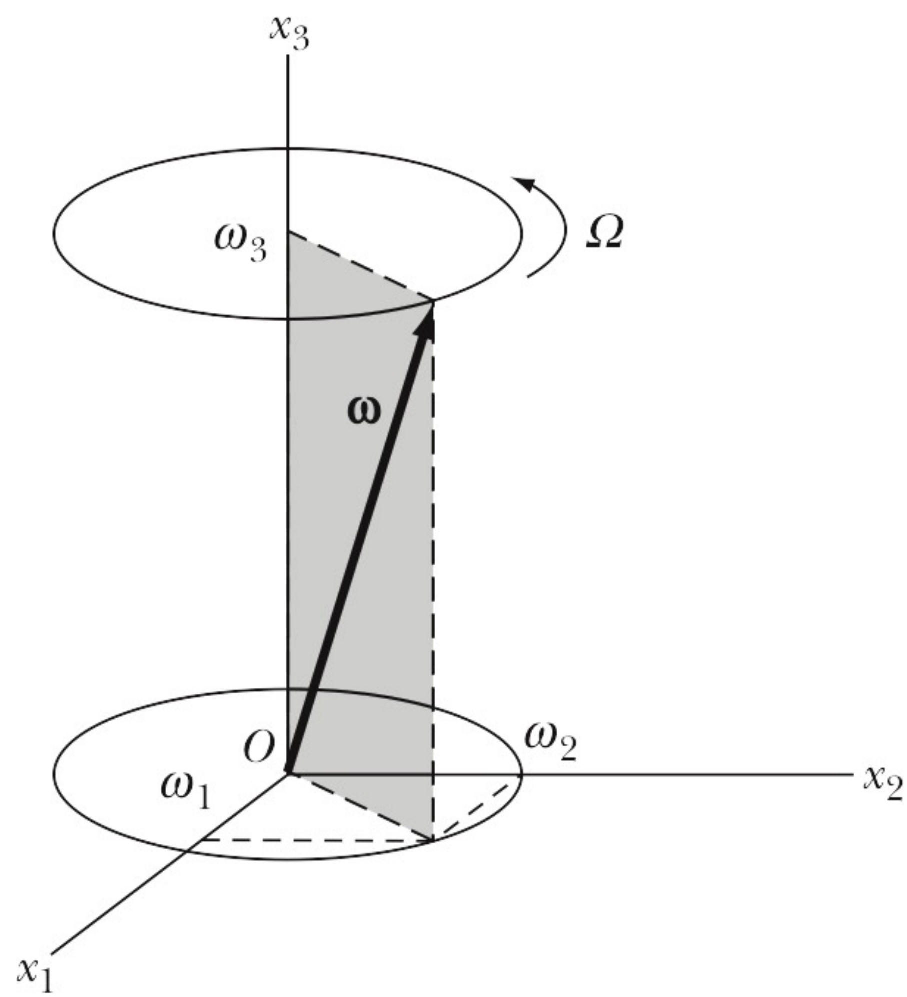
- We thus conclude that

$$\omega_3(t) = \text{constant} = \omega_3$$

- The other two Euler equations:

$$\dot{\omega}_1 = -\left(\frac{I_3 - I_1}{I_1}\omega_3\right)\omega_2 = -\Omega\omega_2$$

$$\dot{\omega}_2 = \left(\frac{I_3 - I_1}{I_1}\omega_3\right)\omega_1 = \Omega\omega_1$$

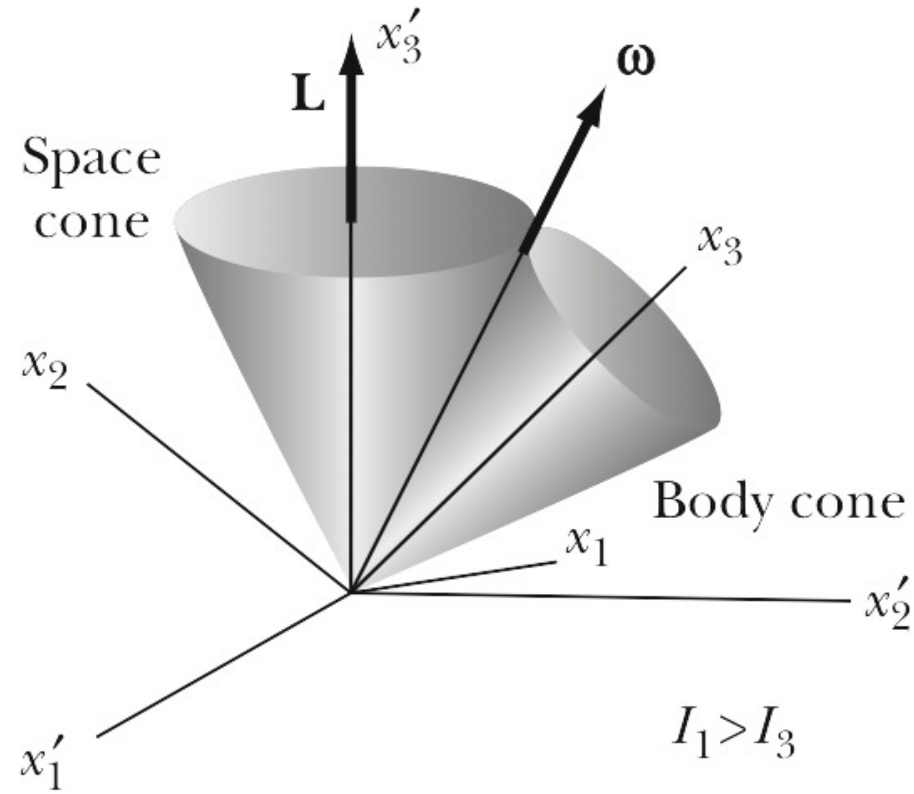


No external forces: angular momentum is conserved.

- Since there are no external forces acting on the system, the angular momentum remains fixed in the fixed reference frame.
- The rotational kinetic energy is also constant.

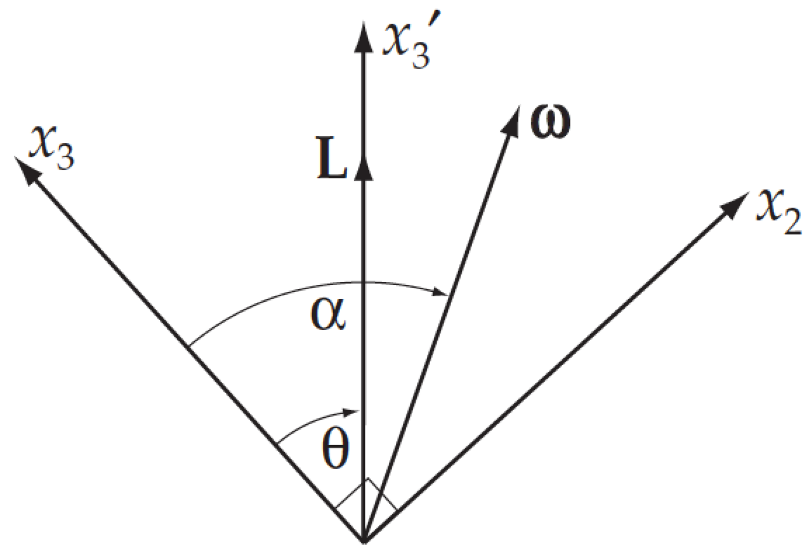
$$T_{rot} = \frac{1}{2}\bar{\omega} \bullet \bar{L}$$

- This requires that the angle between the angular momentum and the angular velocity is constant.

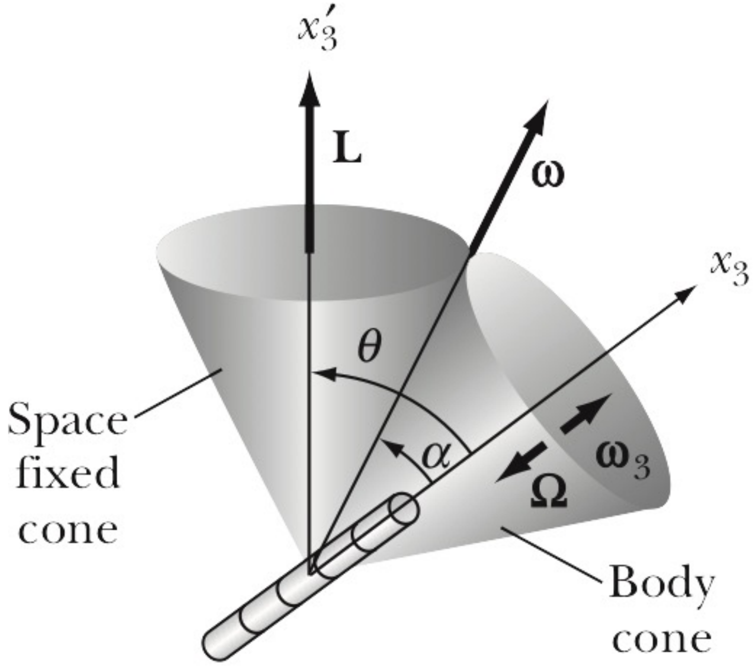


Problem 11.27

A symmetric body moves without the influence of forces or torques. Let x_3 be the symmetry axis of the body and L be along x_3'. The angle between the angular velocity vector and x_3 is α. Let ω and L initially be in the x_2-x_3 plane. What is the angular velocity of the symmetry axis about L in terms of I_1, I_3, ω, and α?

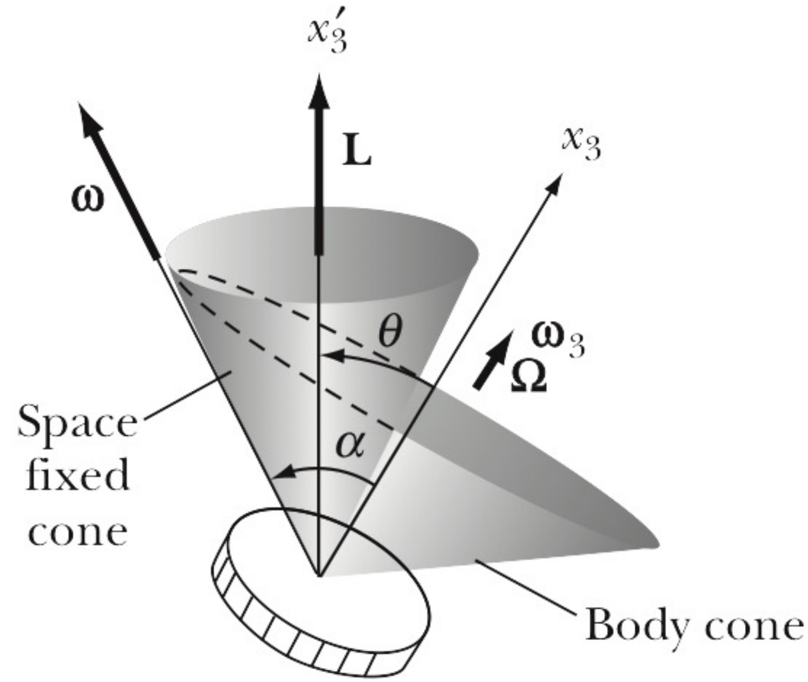


Prolate and Oblate Rotation.



Prolate, $I_1 > I_3$
$\mathbf{\Omega}$, $\boldsymbol{\omega}_3$ have opposite signs.

(a)



Oblate, $I_3 > I_1$
$\mathbf{\Omega}$, $\boldsymbol{\omega}_3$ have same sign.

(b)

External torques: the Euler equations of a rigid body in a force field.

- When the external torque is not equal to 0, the angular momentum of the system is not conserved.
- In this case, we need to use the Euler equations of a rigid body in a force field:

$$N_1 = \frac{dL_1}{dt} + (\bar{\omega} \times \bar{L})_1 = \frac{dL_1}{dt} + (\omega_2 L_3 - \omega_3 L_2) = I_1 \dot{\omega}_1 - (I_2 - I_3)\omega_2 \omega_3$$

$$N_2 = \frac{dL_2}{dt} + (\bar{\omega} \times \bar{L})_2 = \frac{dL_2}{dt} + (\omega_3 L_1 - \omega_1 L_3) = I_2 \dot{\omega}_2 - (I_3 - I_1)\omega_3 \omega_1$$

$$N_3 = \frac{dL_3}{dt} + (\bar{\omega} \times \bar{L})_3 = \frac{dL_3}{dt} + (\omega_1 L_2 - \omega_2 L_1) = I_3 \dot{\omega}_3 - (I_1 - I_2)\omega_1 \omega_2$$

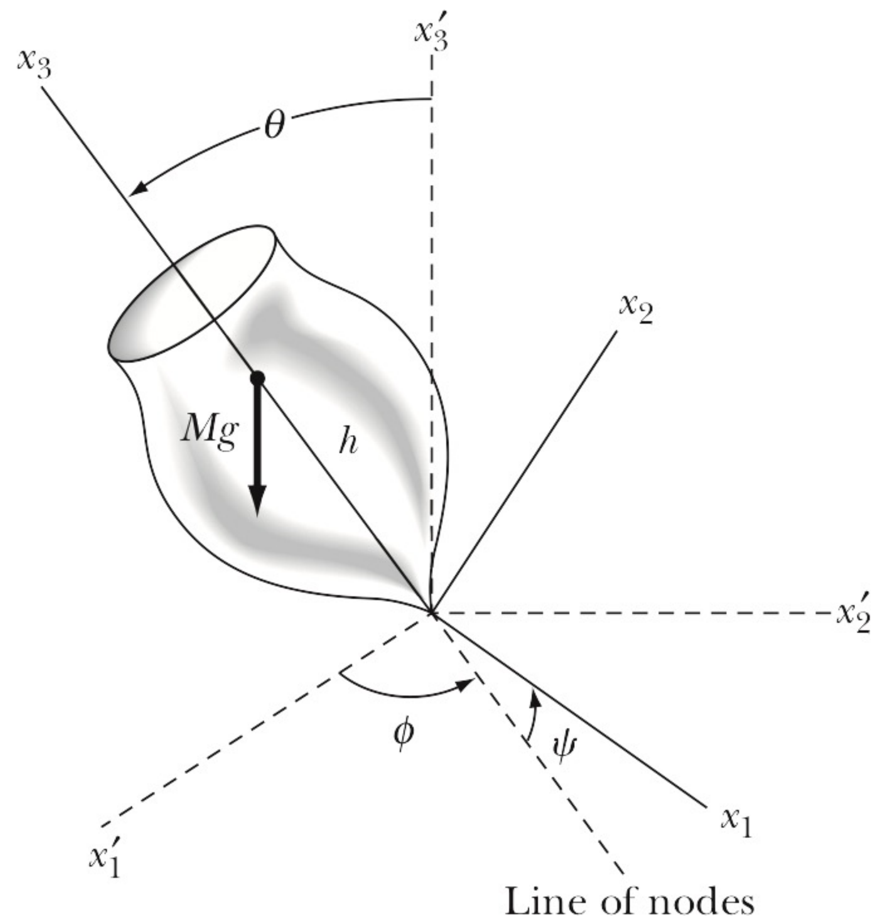
Symmetric top with external torque.

- For this top, $I_1 = I_2$.
- There is a non-zero torque along the x_1 and x_2 axes.
- We conclude:

$$(I_1 - I_2)\omega_1\omega_2 - I_3\dot{\omega}_3 = 0$$

$$-I_3\dot{\omega}_3 = 0$$

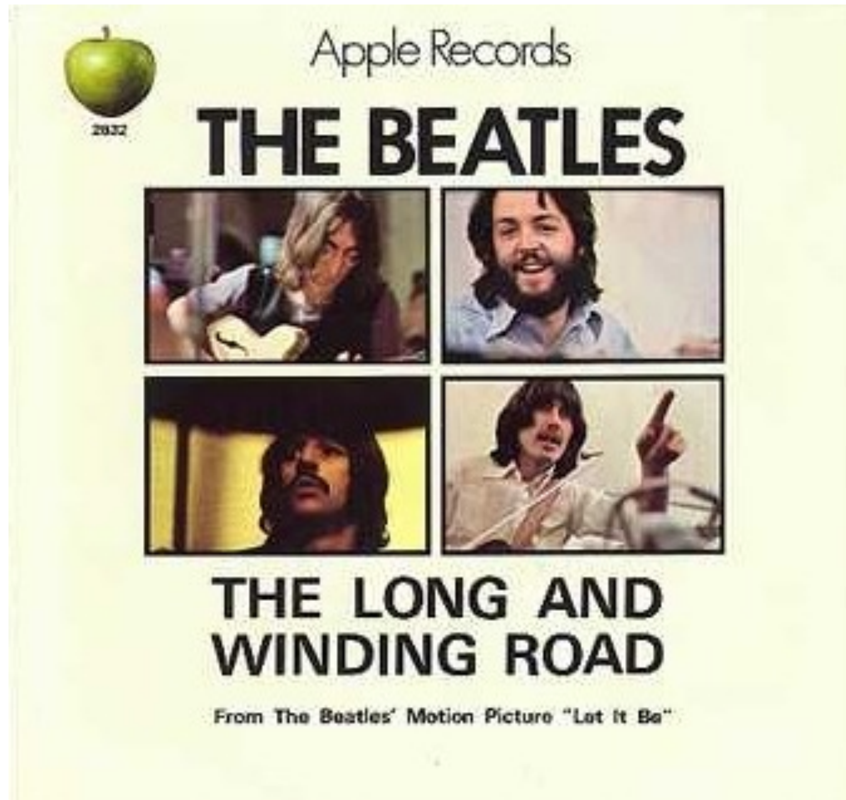
$$\omega_3 = \text{constant}$$



3 Minute 37 Second Intermission.

- Since paying attention for 1 hour and 15 minutes is hard when the topic is physics, let's take a 3 minute 37 second intermission.

- You can:
  - Stretch out.
  - Talk to your neighbors.
  - Ask me a quick question.
  - Enjoy the fantastic music.



Symmetric top with external torque. Effective energy.

- Since no non-conservative forces are acting on the system, energy is conserved.
- Since the angular velocity around the x_3 axis is constant, we can subtract the kinetic energy associated with this motion from the total energy. This produces the ***effective energy E'***:

$$E' = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} \frac{\left(p_\phi - p_\psi \cos\theta\right)^2}{I_1 \sin^2\theta} + Mgh\cos\theta$$

- The momenta are constant and the effective energy only depends on θ and $d\theta/dt$. The problem has been reduced to a one-dimensional problem.

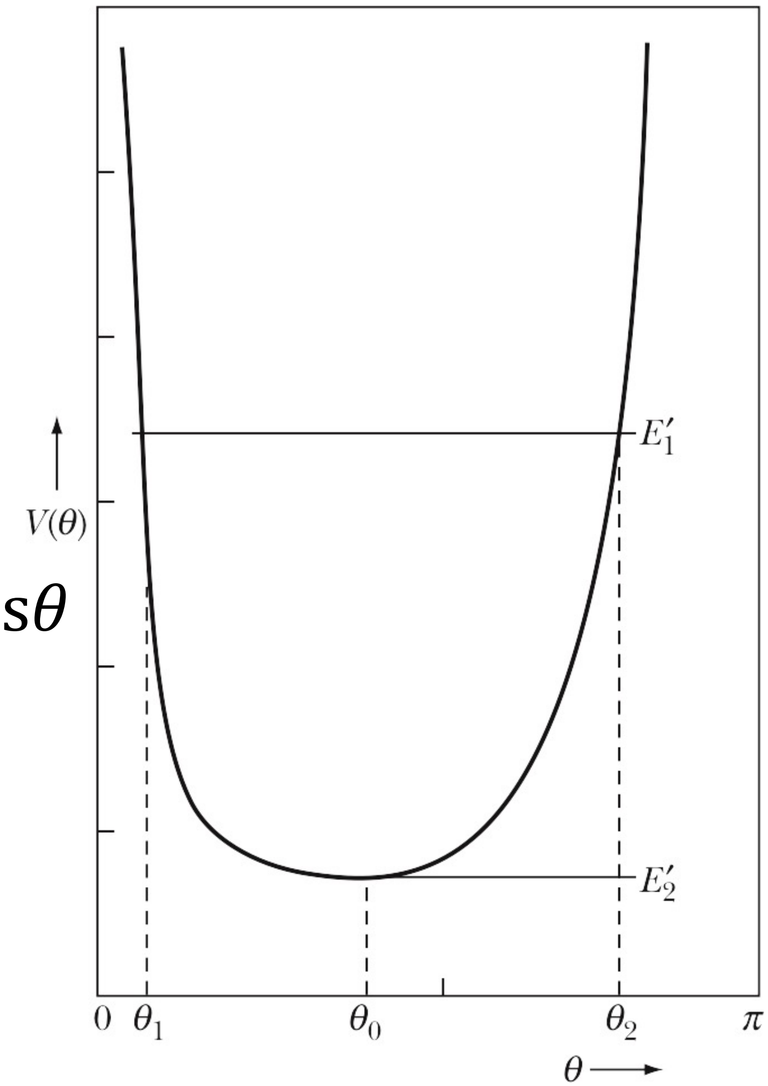
Symmetric top with external torque. Effective potential energy.

- One term of the effective energy depends on $d\theta/dt$, the last two terms depend on θ.

- The effective potential energy is defined as

$$V(\theta) = \frac{1}{2}\frac{\left(p_\phi - p_\psi \cos\theta\right)^2}{I_1 \sin^2\theta} + Mgh\cos\theta$$

- The potential energy shows the limits of motion for a given effective energy.



Symmetric top with external torque.

- The effective energy has a minimum value at a specific angle θ_0.
- At this angle stable precession can be produced if the angular velocity is sufficiently large.

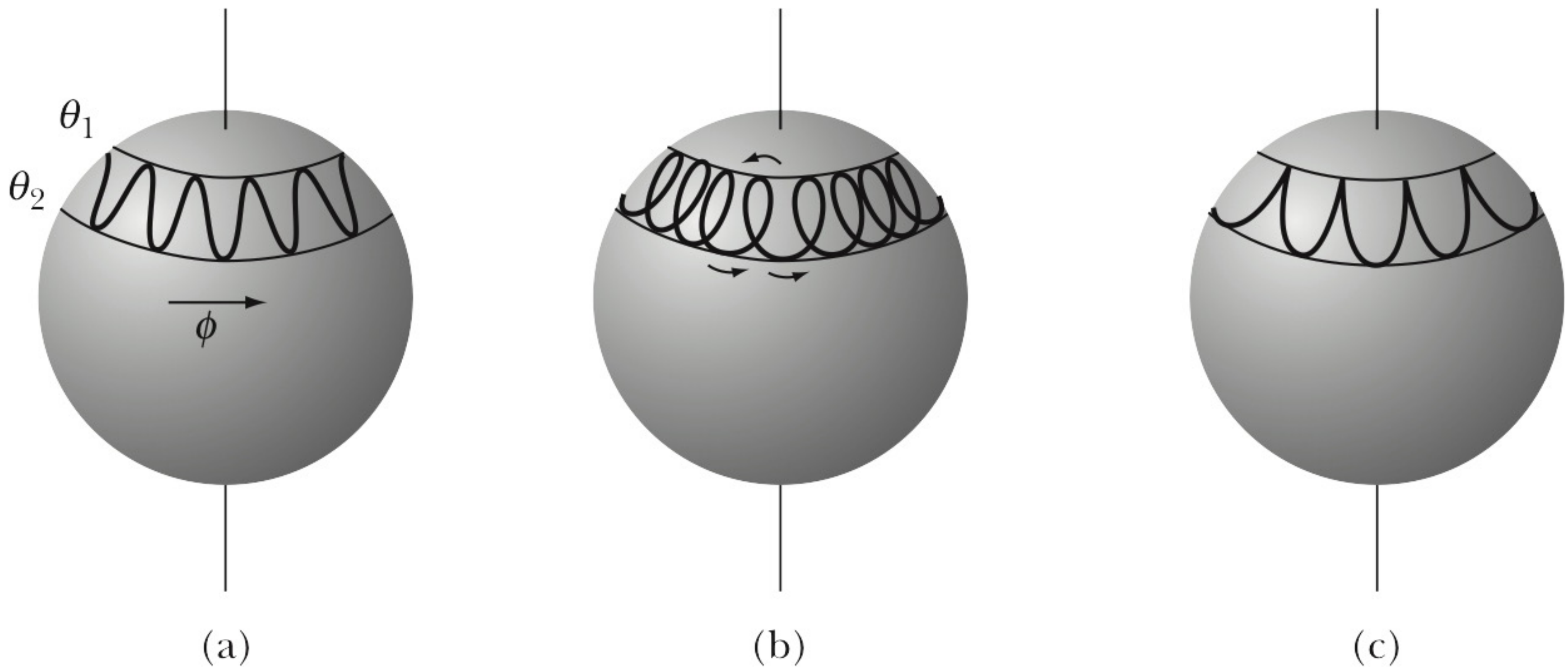
$$\omega_3 \geq \frac{2}{I_3}\sqrt{MghI_1 \cos\theta_0}$$

- In general, there are two precession rates since the there are two possible values of β:

$$\dot{\phi} = \frac{p_\phi - p_\psi \cos\theta_0}{I_1 \sin^2\theta_0} = \frac{\beta}{I_1 \sin^2\theta_0}$$

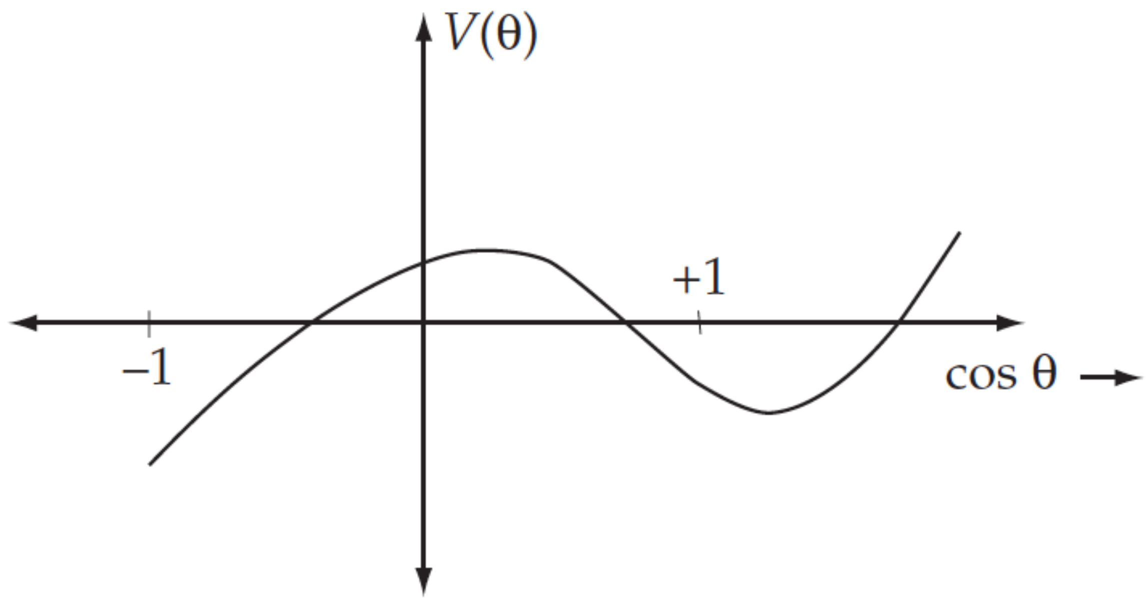
Symmetric top with external torque.

- When the angle of inclination is not θ_0, the system will nutate.



Problem 11.30.

Investigate the equation for the turning points of the nutational motion by setting $\frac{d\theta}{dt} = 0$ in the equation of the effective energy. Show that the resulting equation is cubic in $\cos\theta$ and has two real roots and one imaginary root.



Stability of Rigid-Body Rotations.

- The rotation of a rigid body is stable if the system, when perturbed from its equilibrium condition, carries out small oscillations about it.
- Consider we use the principal axes of rotation to describe the motion, and we choose these axes such that $I_3 > I_2 > I_1$:
  - If the system rotates around the x_1 axis, small perturbations around the x_2 and x_3 axes will cause it to oscillate around the equilibrium values. Rotation around the x_1 axis is stable.
  - Rotation around the x_2 axis is unstable.
  - Rotation around the x_3 axis is stable.

Problem 11.34.

Consider a symmetrical rigid body rotating freely about its center of mass. A frictional torque ($N_f = -b\omega$) acts to slow down the rotation. Find the component of the angular velocity along the symmetry axis as a function of time.

ENOUGH FOR TODAY?