Classical Mechanics
Phy 235, Lecture 19.

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## What a great way to start today!



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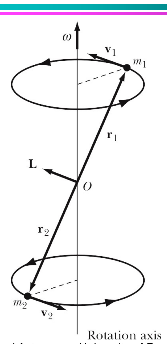
## Just a reminder from Wednesday.

- In general, the angular velocity is not pointed in the same direction as the angular momentum.
- If the angular velocity is directed along the principal axes, the angular velocity and angular momentum are parallel.
- If the object has symmetry, the principal axes are directed along the symmetry axes of the object.



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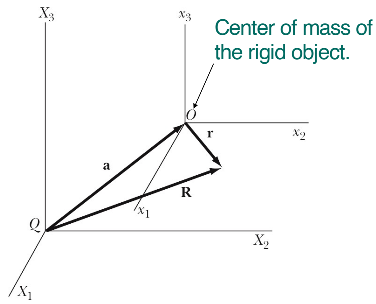
## Transformations of the Inertia Tensor. Connecting Inertia Tensors (*I* and *J*).

A translation.



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## Connecting Inertia Tensors (*I* and *J*). The Steiner's Parallel-Axis Theorem.

- In the displaced reference frame (frame *X*):

$$J_{ij} = \sum_{\alpha} m_\alpha \left( \delta_{ij} \sum_k X_{\alpha,k}^2 - X_{\alpha,j} \right)$$

- This inertia tensor is related to the inertia tensor in the reference frame where the origin is located at the center-of-mass of the object:

$$J_{ij} = I_{ij} + \sum_{\alpha} m_\alpha \left( \delta_{ij} \sum_k a_k^2 - a_i a_j \right) = I_{ij} + M\left(\delta_{ij} a^2 - a_i a_j\right)$$



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## Connecting Inertia Tensors. Rotations.

- Rotations can be expressed in terms of a rotation matrix $\lambda_{ij}$:

$$x_i = \sum_j \lambda_{ji} x_j{}'$$

- The transformation of the inertia tensors can be written as

$$I'_{im} = \sum_{k,l} \lambda_{ik} \lambda_{ml} I_{kl} = \sum_{k,l} \lambda_{ik} I_{kl} \lambda_{lm}^t$$

Transposed matrix element

or

$$\{I'\} = \{\lambda\}\{I\}\{\lambda^t\}$$



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## Problem 11.16.

Consider the following inertia tensor:

$$\{I\} = \begin{Bmatrix} \frac{1}{2}(A+B) & \frac{1}{2}(A-B) & 0 \\ \frac{1}{2}(A-B) & \frac{1}{2}(A+B) & 0 \\ 0 & 0 & C \end{Bmatrix}$$

Perform a rotation of the coordinate system by an angle $\theta$ about the $x_3$ axis. Evaluate the transformed tensor elements, and show that the choice $\theta = \pi/4$ renders the inertia tensor diagonal with elements $A$, $B$, and $C$.



## Problem 11.16.

- The rotation matrix is

$$\{\lambda\} = \begin{pmatrix} cos\theta & sin\theta & 0 \\ -sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- The transformed inertia tensor is thus

$$(\mathrm{I}') = (\lambda)(\mathrm{I})(\lambda^t) =$$

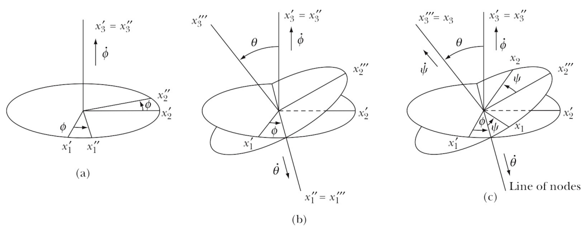
$$= \begin{bmatrix} \frac{1}{2}(A+B)+(A-B)\cos\theta\sin\theta & \frac{1}{2}(A-B)\cos^2\theta-\frac{1}{2}(A-B)\sin^2\theta & 0 \\ -\frac{1}{2}(A-B)\sin^2\theta+\frac{1}{2}(A-B)\cos^2\theta & \frac{1}{2}(A+B)-(A-B)\cos\theta\sin\theta & 0 \\ 0 & 0 & C \end{bmatrix}$$



## 2 Minute 41 Second Intermission.

- Since paying attention for 1 hour and 15 minutes is hard when the topic is physics, let's take a 2 minute 41 second intermission.
- You can:
  - Stretch out.
  - Talk to your neighbors.
  - Ask me a quick question.
  - Enjoy the fantastic music.

# Euler Angles.






# Euler Angles. Transformation Matrix.

Rotation around $x_3'''$ axis.

Rotation around $x_1''$ axis.

Rotation around $x_3'$ axis.

$$\lambda = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$



# Euler Angles. Angular Velocity.

$$\bar{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} \dot{\phi}_1 + \dot{\theta}_1 + \dot{\psi}_1 \\ \dot{\phi}_2 + \dot{\theta}_2 + \dot{\psi}_2 \\ \dot{\phi}_3 + \dot{\theta}_3 + \dot{\psi}_3 \end{pmatrix} = \begin{pmatrix} \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \dot{\phi}\cos\theta + \dot{\psi} \end{pmatrix}$$

## Lagrange's equations for the three Euler angles.

- We can obtain a Lagrange's equation for each Euler angle:
  - $\phi$:
    $$\frac{d}{dt}\left\{I_1\omega_1 \sin\theta \sin\psi + I_2\omega_2 \sin\theta \cos\psi + I_3\omega_3 \cos\theta\right\} = 0$$
  - $\theta$:
    $$\dot{\phi}\left(\left\{I_1\omega_1 \sin\psi + I_2\omega_2 \cos\psi\right\}\cos\theta - I_3\omega_3 \sin\theta\right) -$$
    $$\frac{d}{dt}\left\{I_1\omega_1 \cos\psi - I_2\omega_2 \sin\psi\right\} = 0$$
  - $\psi$:
    $$\left(I_1 - I_2\right)\omega_1\omega_2 - I_3\dot{\omega}_3 = 0 \longleftarrow$$
    Only equation that contains just angular velocities.



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## Lagrange's equations for the three Euler angles.

- Since our choice of coordinate axes was arbitrary, we can find the following relations for the three components of the angular velocity:

$$\left(I_1 - I_2\right)\omega_1\omega_2 - I_3\dot{\omega}_3 = 0$$

$$\left(I_2 - I_3\right)\omega_2\omega_3 - I_1\dot{\omega}_1 = 0$$

$$\left(I_3 - I_1\right)\omega_3\omega_1 - I_2\dot{\omega}_2 = 0$$



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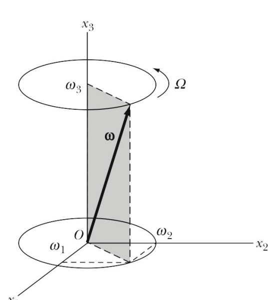
## Example: symmetric top.

- Two different principle moments: $I_1 = I_2$ and $I_3$.
- One of the Euler equations tells us:
  $$I_3\dot{\omega}_3 = 0$$
- We thus conclude that
  $$\omega_3(t) = \text{constant} = \omega_3$$
- The other two Euler equations:
  $$\dot{\omega}_1 = -\left(\frac{I_3 - I_1}{I_1}\omega_3\right)\omega_2 = -\Omega\omega_2$$
  $$\dot{\omega}_2 = \left(\frac{I_3 - I_1}{I_1}\omega_3\right)\omega_1 = \Omega\omega_1$$



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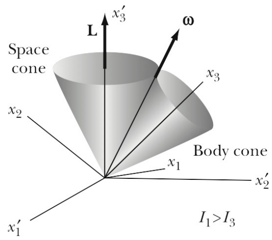
## No external forces: angular momentum is conserved.

- Since there are no external forces acting on the system, the angular momentum remains fixed in the fixed reference frame.
- The rotational kinetic energy is also constant.

$$T_{rot} = \frac{1}{2}\bar{\omega} \bullet \bar{L}$$

- This requires that the angle between the angular momentum and the angular velocity is constant.

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## ENOUGH FOR TODAY?

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