Classical Mechanics
Phy 235, Lecture 18.

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A whale saved a train in Spijkenisse.

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Check the grading matrix before you submit your final paper.

Grading matrix

______ 10 points: Overall layout. Are all components there?
______ 10 points: Figures labelled and with caption.
______ 10 points: Proper references.
______ 20 points: Clarity of writing.
______ 20 points: Formulation of problem.
______ 20 points: Analysis of problem.
______ 10 points: Summary.

Loss of points:

______ 25 points: No discussion with writing fellows.
______ 25 points: No draft pdf version submitted.
______ 25 points: Post deadline submission.

Total points:

______

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Rigid Bodies

- Rigid body:
  - Collection of particles with fixed relative positions, independent of the motion carried out by the body.
- The following equations describe the moment of inertia, the angular momentum, and the kinetic energy of a rotating rigid body when the motion of the rigid body can be reduced to two-dimensional motion:

$$I = \sum_i m_i r_i^2$$

$$\vec{L} = I\vec{\omega}$$

$$T = \frac{1}{2} I \omega^2$$

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The Inertia Tensor. Describing 3D motion.

- We assume that the rotating reference frame that can be used to describe the rotation of the object is fixed to the rigid body. In this case, each component of the object will be at rest in the rotating frame.

$$\vec{v}_f = \vec{V} + \vec{v}_r + \vec{\omega}\times\vec{r} = \vec{V} + \vec{\omega}\times\vec{r}$$

$$T = \tfrac{1}{2}\sum_\alpha m_\alpha \left(V^2 + 2\vec{V}\cdot\{\vec{\omega}\times\vec{r}_\alpha\} + \{\vec{\omega}\times\vec{r}_\alpha\}\cdot\{\vec{\omega}\times\vec{r}_\alpha\}\right)$$

- The kinetic energy of the rigid body will be equal to

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The Inertia Tensor

- The kinetic energy of the rigid body can be rewritten as

$$T = \frac{1}{2}MV^2 + \frac{1}{2}\sum_{i,j} I_{ij}\omega_i\omega_j = T_{CM} + T_{rot}$$

- This equation includes the inertia tensor:

$$\{I\} = \begin{pmatrix} \sum_\alpha m_\alpha \left(r_{\alpha,2}{}^2 + r_{\alpha,3}{}^2\right) & -\sum_\alpha m_\alpha r_{\alpha,1} r_{\alpha,2} & -\sum_\alpha m_\alpha r_{\alpha,1} r_{\alpha,3} \\ -\sum_\alpha m_\alpha r_{\alpha,2} r_{\alpha,1} & \sum_\alpha m_\alpha \left(r_{\alpha,1}{}^2 + r_{\alpha,3}{}^2\right) & -\sum_\alpha m_\alpha r_{\alpha,2} r_{\alpha,3} \\ -\sum_\alpha m_\alpha r_{\alpha,3} r_{\alpha,1} & -\sum_\alpha m_\alpha r_{\alpha,3} r_{\alpha,2} & \sum_\alpha m_\alpha \left(r_{\alpha,1}{}^2 + r_{\alpha,2}{}^2\right) \end{pmatrix}$$

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Properties of the Inertia Tensor

- The tensor is symmetric: $I_{ij} = I_{ji}$. Of the 9 parameters, only 6 are free parameters.
- The non-diagonal tensor elements are called **products of inertia.**
- The diagonal tensor elements are the moments of inertia with respect to the three coordinate axes of the rotating frame.

Angular Momentum.

- The total angular momentum of a rotating rigid object is the vector sum of each component of the object:

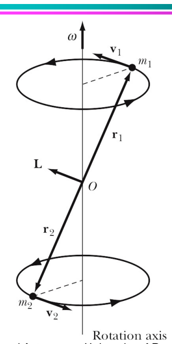
$$L_i = \sum_\alpha (\vec{r}_\alpha \times \vec{p}_\alpha)_i = \sum_\alpha (\vec{r}_\alpha \times m_\alpha (\vec{\omega} \times \vec{r}_\alpha))_i = \sum_j I_{ij}\omega_j$$

- The total kinetic energy of the system can also be expressed in terms of the angular momentum:

$$T_{rot} = \frac{1}{2}\sum_{i,j} I_{ij}\omega_i\omega_j = \frac{1}{2}\sum_i \omega_i \left(\sum_j I_{ij}\omega_j\right) = \frac{1}{2}(\vec{\omega} \cdot \vec{L})$$

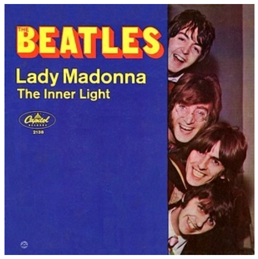
Angular Momentum.

- The angular velocity of an object does not have to be parallel to the angular momentum of the object.



2 Minute 18 Second Intermission.

- Since paying attention for 1 hour and 15 minutes is hard when the topic is physics, let's take a 2 minute 18 second intermission.
- You can:
  - Stretch out.
  - Talk to your neighbors.
  - Ask me a quick question.
  - Enjoy the fantastic music.



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Simplifying your life!

- We can simplify the solutions if we can choose our coordinate axes such that the inertia tensor only has diagonal elements (**principal axes of inertia**):

$$\{I\} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

- Many calculations simplify under these conditions:

$$L_i = I_i \omega_i$$

$$T_{rot} = \frac{1}{2}\sum_i I_i \omega_i^2$$

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How do we find the principal axes of inertia?

- Require:

$$(I_{11} - I)\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 = 0$$
$$I_{21} + (I_{22} - I)\omega_2 + I_{23}\omega_3 = 0$$
$$I_{31}\omega_1 + I_{32}\omega_2 + (I_{33} - I)\omega_3 = 0$$

- Solutions only exist if:

$$\begin{vmatrix} I_{11} - I & I_{12} & I_{13} \\ I_{21} & I_{22} - I & I_{23} \\ I_{31} & I_{32} & I_{33} - I \end{vmatrix} = 0$$

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Problem 11.13.

A three-particle system consists of masses m_i and coordinates (x_1, x_2, x_3) as follows:

$$\begin{aligned} m_1 &= 3m \quad (b, 0, b) \\ m_2 &= 4m \quad (b, b, -b) \\ m_3 &= 2m \quad (-b, b, 0) \end{aligned}$$

Find the inertia tensor, the principal axes, and the principal moments of inertia.

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Solution Problem 11.13.

- The inertia tensor is equal to

$$\{I\} = mb^2 \begin{vmatrix} 13 & -2 & 1 \\ -2 & 16 & 4 \\ 1 & 4 & 15 \end{vmatrix}$$

- The principal moments of inertia can be found by solving:

$$mb^2 \begin{vmatrix} 13-\lambda & -2 & 1 \\ -2 & 16-\lambda & 4 \\ 1 & 4 & 15-\lambda \end{vmatrix} = 0$$

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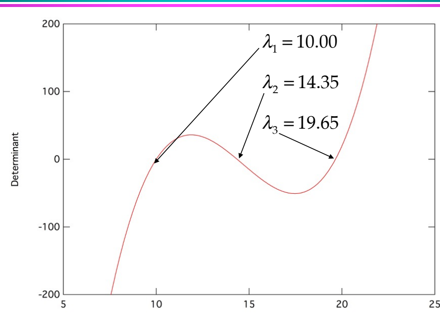
Solution Problem 11.13.
Analytical solutions do not always exist.



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Solution Problem 11.13.
Finding the principal axes of inertia.

- To find the principal axes of inertia we have to determine for which rotational velocities the following relation holds:

$$\{I\}\vec{\omega}_i = \lambda_i \vec{\omega}_i$$

- Solving these three equations (for i = 1, 2, and 3) provides us with the three principal axes of inertia.
- We note that these three axes are mutually orthogonal.

ENOUGH FOR TODAY?