Classical Mechanics
Phy 235, Lecture 06.

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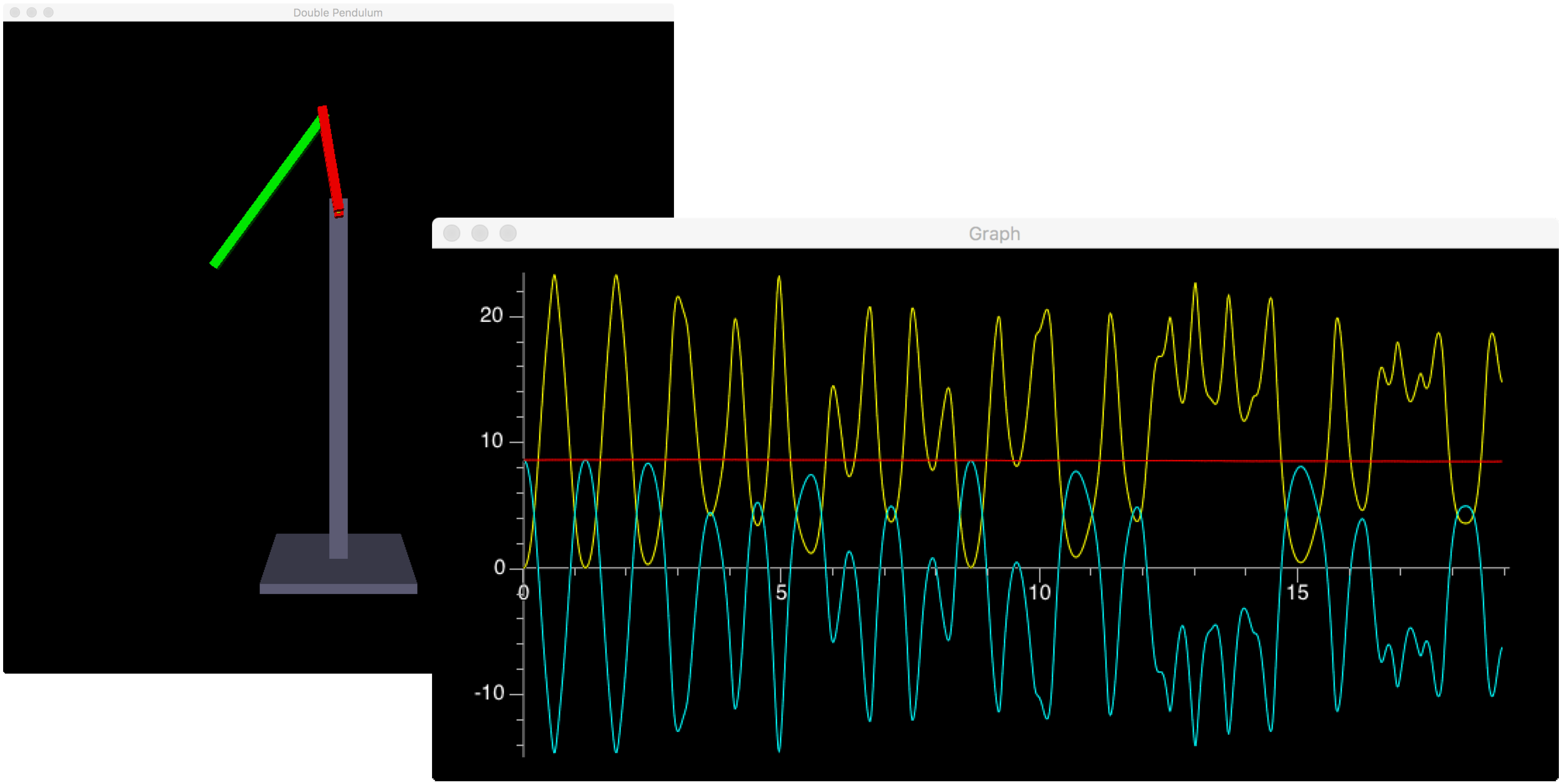
A few remarks …

- Midterm exam # 1:
  - Next week on Tuesday 9/23, 8.00 – 9.20 am, B&L 109:
  - Chapters 1 – 4.
  - Equation sheet will be provided.
- Exam material will be reviewed on Wednesday 9/17 during lecture.
- There will be extra office hours next week on Monday 9/22.
- There will be:
  - No office hours on Wednesday 9/24 and Thursday 9/25.
  - No recitations on Tuesday 9/23 and Thursday 9/25.
  - No regular homework due on Friday 9/26. But the extra credit homework is due on that day.

Chaos.
Simple systems can be chaotic.



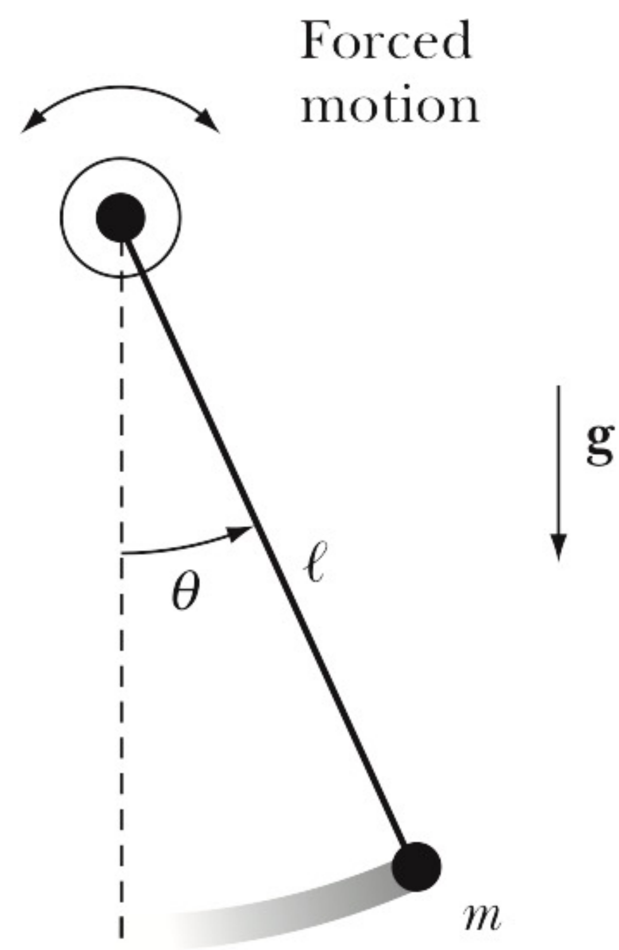
http://www.glowscript.org/#/user/wolfs/folder/Public/program/DoublePendulum

Chaos.

- Consider a simple driven pendulum.
- The motion of the driven pendulum can be written as

$$\ddot{\theta} = -\frac{b}{ml^2}\dot{\theta} - \frac{g}{l}\sin(\theta) + \frac{N_d}{ml^2}\cos(\omega t)$$

- Even when the driving force is a harmonic driving force, the resulting motion may be chaotic.
- How do we identify chaos?
- All will be revealed today.



Studying the solution with Mathematica.

```
(* Set the values of the various parameters *)
c = 0.2;
w = 0.694;
F = 0.52;
pi = N[Pi];
cycles = 50;
steps = 30;
```

Initial parameters.

Note: “lhs := rhs /; test”. Use this definition only if test = True.

```
(* Solve the differential equations with the given set of initial conditions. *)
sol = NDSolve[
   {x'[t] == v[t],
    v'[t] == -c*v[t] - Sin[x[t]] + F*Cos[w*t],
    x[0] == 0.8,
    v[0] == 0.8},
   {x, v}, {t, 0, (cycles*(2 Pi) / w)}, MaxSteps → 20000];
```

Diff. equation.

Boundary conditions (*x* and *v* at $t = 0$).

```
(* Create a function "reduce" that translate all angles back to the region between -Pi and +Pi *)
reduce[x_] := Mod[x, 2 pi] /; Mod[x, 2 pi] <= pi;
reduce[x_] := (Mod[x, 2 pi] - 2 pi) /; Mod[x, 2 pi] > pi;
```

Mod function returns the result of the remainder of the division by 2π..

Creating plots to look at the solution.

Plot angular velocity vs time.

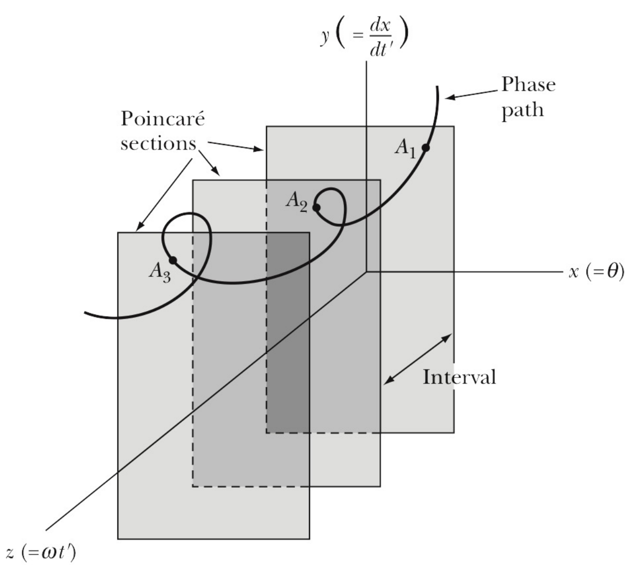
```
(* Plot the solution of the differential equations: v[t] vs t. *)
ParametricPlot[
  Evaluate[{t, v[t]} /. sol], {t, 0, (cycles * (2 Pi) / w)},
  PlotRange → {{((cycles - 20) * (2 Pi) / w), (cycles * (2 Pi) / w)}, {-3, 3}},
  AxesLabel → {"t/t0", "v (rad/s)"}
 ];
```

Plot angular velocity vs angle.

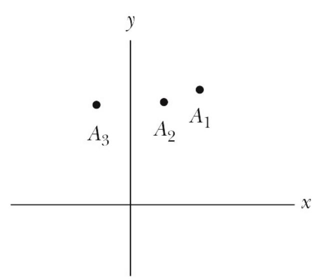
```
(* Plot the phase diagram. *)
ParametricPlot[
  Evaluate[{x[t], v[t]} /. sol], {t, ((cycles - 20) * (2 Pi) / w), (cycles * (2 Pi) / w)},
  PlotRange → {-3, 3},
  AxesLabel → {"x (rad)", "v (rad/s)"}
 ];
```

Note: the first 20 cycle are skipped.

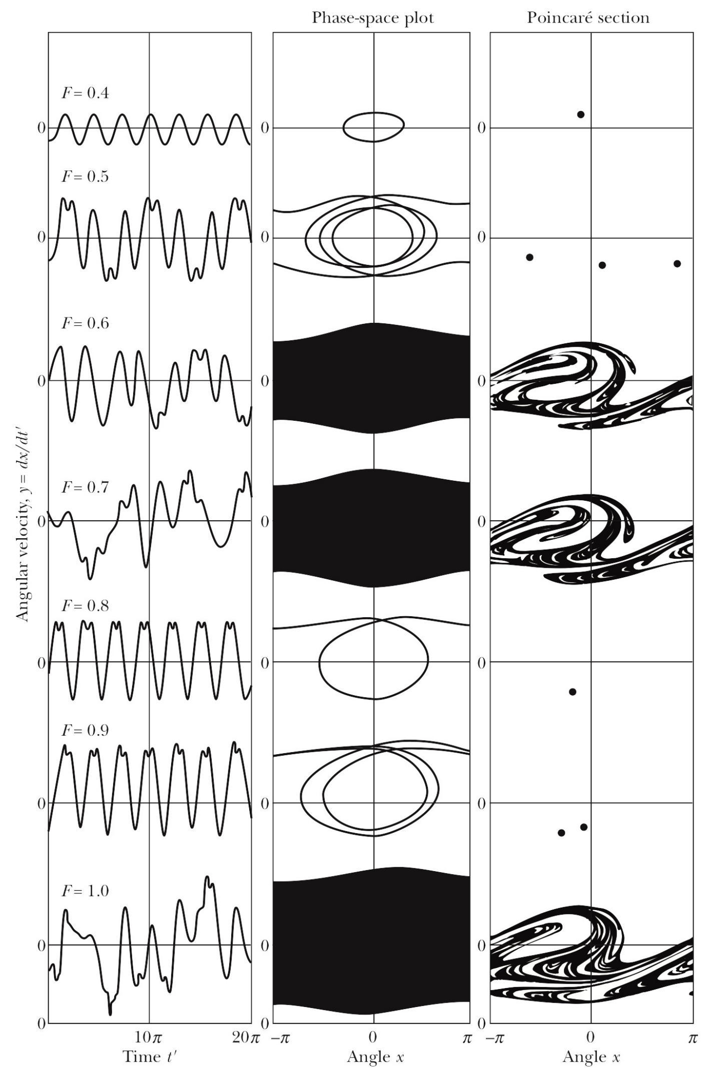
Looking at chaos.
Poincare plots.



(a)



(b)



SHM

Periodic

Chaotic

Chaotic

"SHM"

Periodic

Chaotic

3 Minute 47 Second Intermission.

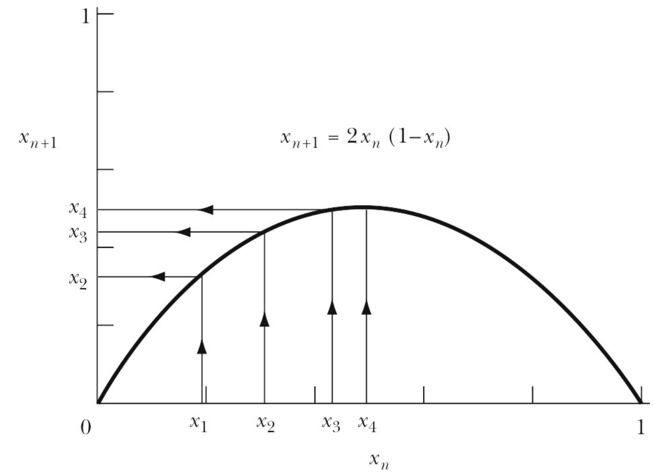
- Since paying attention for 1 hour and 15 minutes is hard when the topic is physics, let's take a 3 minute 47 second intermission.
- You can:
  - Stretch out.
  - Talk to your neighbors.
  - Ask me a quick question.
  - Enjoy the fantastic music.

Logistic equations.
Creating chaos with maps.

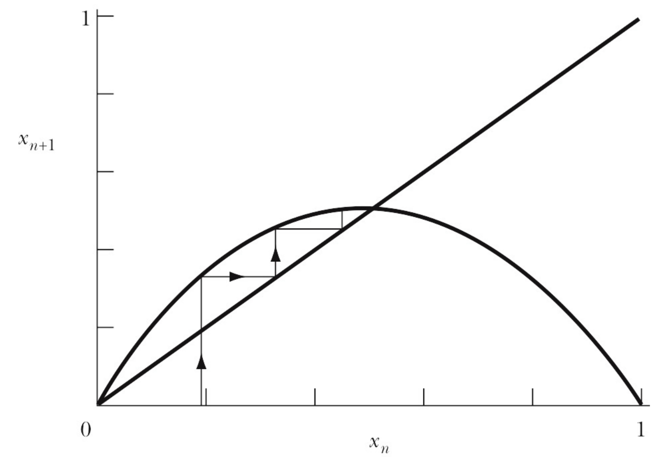
- Assume the position parameter of a system at time t_{N+1}, x_{N+1}, depends on the value of the position at time t_N:

$$x_{N+1} = \alpha x_N(1 - x_N) = f(\alpha, x_N)$$

- For certain values of α, the system will evolve towards a stable equilibrium. For other starting values, the system may evolve towards more than one possible solution.

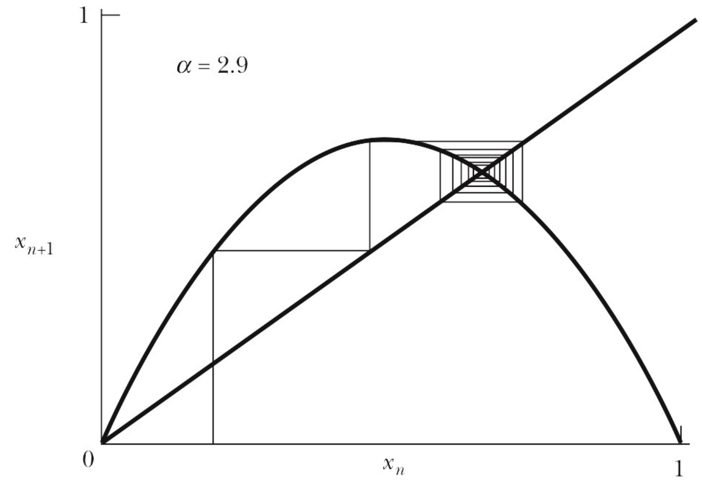


(a)

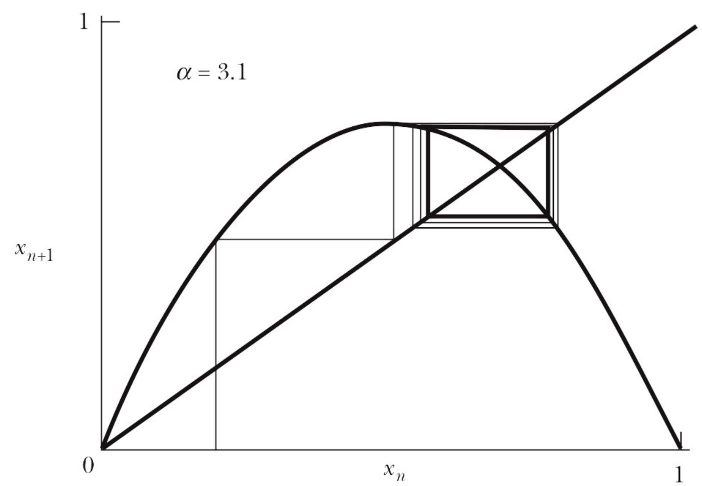


(b)

Logistic equations.
Creating chaos with maps.



(a)



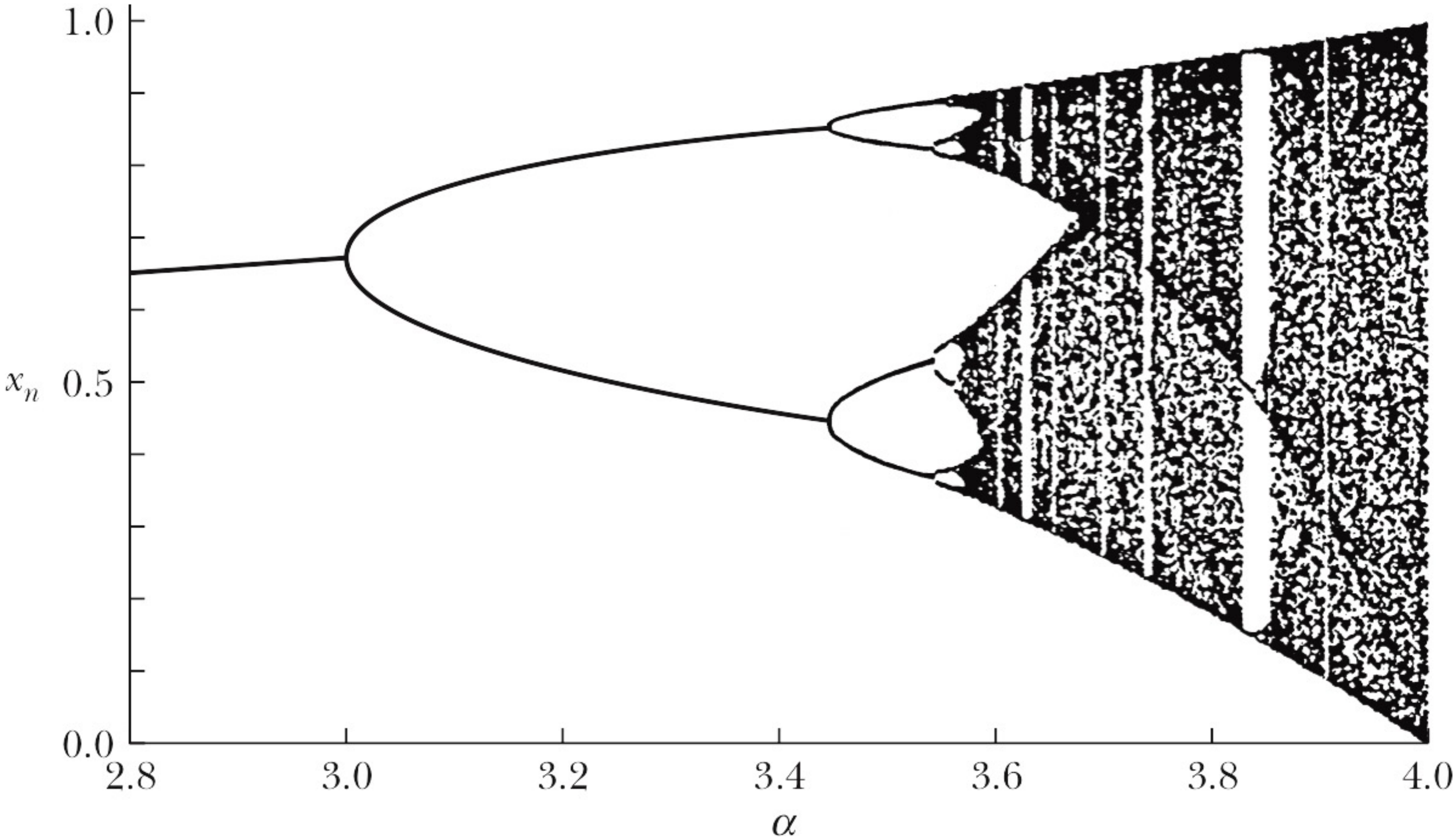
(b)

Logistic Maps

- Using tools such as VPython, it is easy to explore logistic maps.
- Let us have a look:

  https://www.glowscript.org/#/user/wolfs/folder/Public/program/PHY235-LogisticMaps

Bifurcation diagrams.
A view of possible modes of evolution.



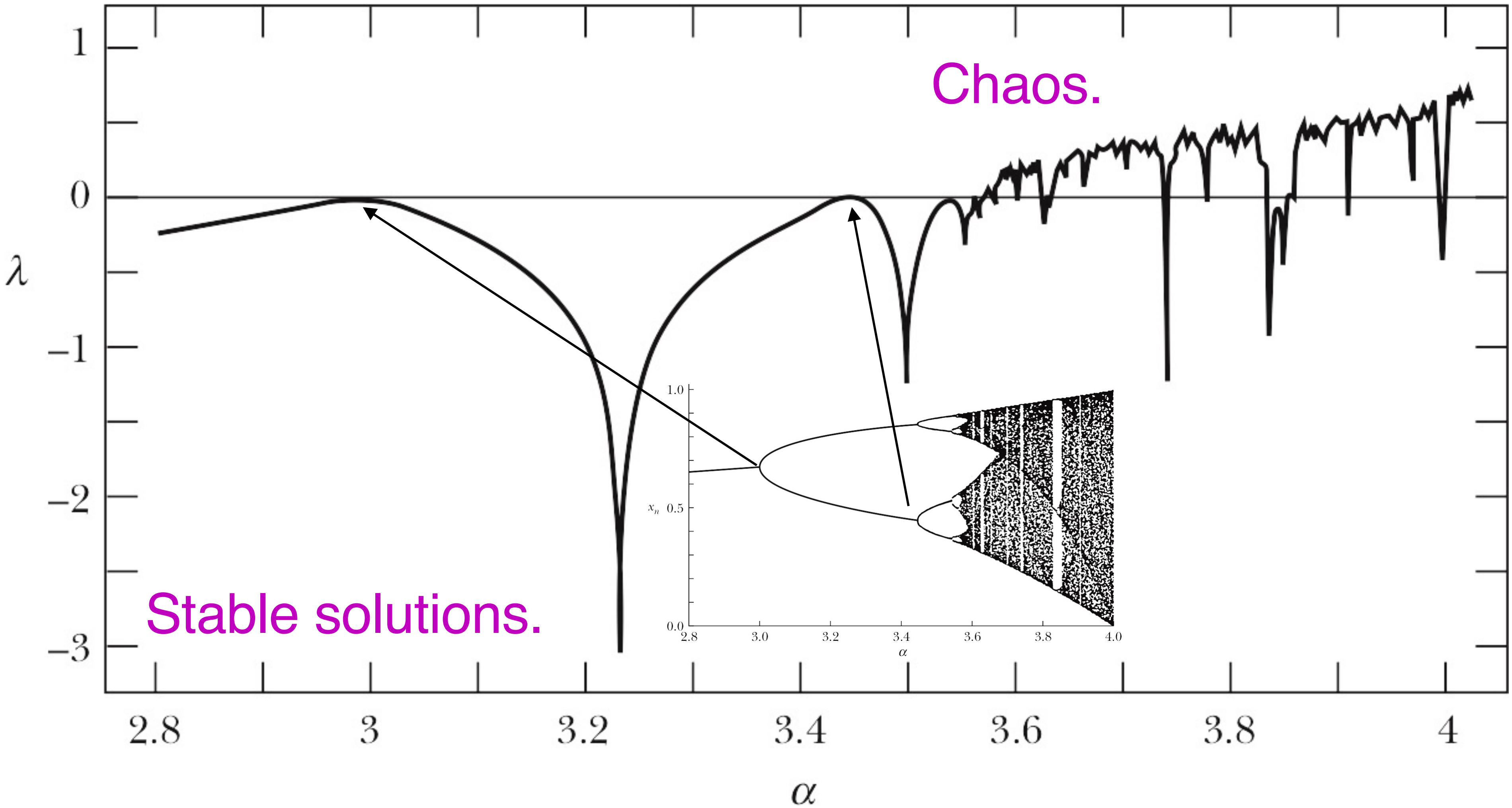
The Lyapunov exponent λ.

- The development of chaos can be studied by examining the Lyapunov exponent λ:

$$\lambda = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n-1}\left|\frac{df(\alpha, x_i)}{dx}\right|\right)$$

- This exponent is a measure of the difference between solutions when we make a small change in the initial conditions:
    - If $\lambda < 0$: stable solutions.
    - If $\lambda = 0$: doubling of the number of solutions
    - If $\lambda > 0$: chaos.

The Lyapunov exponent λ.



This completes the material that is going to be covered on Midterm Exam # 1.

ENOUGH FOR TODAY?