

11 Dynamics of Rigid Bodies

A rigid body is a collection of particles with fixed relative positions, independent of the motion carried out by the body. The dynamics of a rigid body has been discussed in our introductory courses, and the techniques discussed in these courses allow us to solve many problems in which the motion can be reduced to two-dimensional motion. In this special case, we found that the angular momentum associated with the rotation of the rigid object is directed in the same direction as the angular velocity:

$$\vec{L} = I\vec{\omega} \quad (11.1)$$

In this equation, I is the moment of inertia of the rigid body which was defined as

$$I = \sum_i m_i r_i^2 \quad (11.2)$$

where r_i is the distance of mass m_i from the rotation axis. We also found that the kinetic energy of the body, associated with its rotation, is equal to

$$T = \frac{1}{2}I\omega^2 \quad (11.3)$$

The complexity of the motion increases when we need three dimensions to describe the motion. There are many different ways to describe motion in three dimensions. One common method is to describe the motion of the center of mass (in a fixed coordinate system) and to describe the motion of the components around the center of mass (in the rotating coordinate system).

11.1 The Inertia Tensor

In Chapter 10 we derived the following relation between the velocity of a particle in the fixed reference frame, v_f , and its velocity in the rotating reference frame v_r :

$$\vec{v}_f = \vec{V} + \vec{v}_r + \vec{\omega} \times \vec{r} \quad (11.4)$$

If we assume that the rotating frame is fixed to the rigid body, then $v_r = 0$.

The total kinetic energy of the rigid body is the sum of the kinetic energies of each component of the rigid body. Thus

$$\begin{aligned} T &= \sum_{\alpha} \left\{ \frac{1}{2} m_{\alpha} v_{\alpha}^2 \right\} = \frac{1}{2} \sum_{\alpha} m_{\alpha} \left(\left\{ \vec{V} + \vec{\omega} \times \vec{r}_{\alpha} \right\} \cdot \left\{ \vec{V} + \vec{\omega} \times \vec{r}_{\alpha} \right\} \right) = \\ &= \frac{1}{2} \sum_{\alpha} m_{\alpha} \left(V^2 + 2\vec{V} \cdot \left\{ \vec{\omega} \times \vec{r}_{\alpha} \right\} + \left\{ \vec{\omega} \times \vec{r}_{\alpha} \right\} \cdot \left\{ \vec{\omega} \times \vec{r}_{\alpha} \right\} \right) \end{aligned} \quad (11.5)$$

Let us now examine the three terms in this expression:

$$\frac{1}{2} \sum_{\alpha} m_{\alpha} V^2 = \frac{1}{2} V^2 \sum_{\alpha} m_{\alpha} = \frac{1}{2} M V^2 \quad (11.6)$$

$$\frac{1}{2} \sum_{\alpha} m_{\alpha} \left(2\vec{V} \cdot \left\{ \vec{\omega} \times \vec{r}_{\alpha} \right\} \right) = \vec{V} \cdot \left\{ \vec{\omega} \times \sum_{\alpha} \left(m_{\alpha} \vec{r}_{\alpha} \right) \right\} = \vec{V} \cdot \left\{ \vec{\omega} \times \left(M\vec{R} \right) \right\} = 0 \quad (11.7)$$

$$\begin{aligned} \frac{1}{2} \sum_{\alpha} m_{\alpha} \left(\left\{ \vec{\omega} \times \vec{r}_{\alpha} \right\} \cdot \left\{ \vec{\omega} \times \vec{r}_{\alpha} \right\} \right) &= \frac{1}{2} \sum_{\alpha} m_{\alpha} \left(\omega^2 r_{\alpha}^2 - \left\{ \vec{\omega} \cdot \vec{r}_{\alpha} \right\}^2 \right) = \\ &= \frac{1}{2} \sum_{\alpha} m_{\alpha} \left(\left[\sum_i \omega_i^2 \right] \left[\sum_k r_{\alpha,k}^2 \right] - \left[\sum_i \left(\omega_i r_{\alpha,i} \right) \right] \left[\sum_j \left(\omega_j r_{\alpha,j} \right) \right] \right) = \\ &= \frac{1}{2} \sum_{i,j} \left\{ \sum_{\alpha} m_{\alpha} \left(\delta_{ij} \left[\sum_k r_{\alpha,k}^2 \right] - r_{\alpha,i} r_{\alpha,j} \right) \right\} \omega_i \omega_j = \frac{1}{2} \sum_{i,j} I_{ij} \omega_i \omega_j \end{aligned} \quad (11.8)$$

The second term is zero, if we choose the origin of the rotating coordinate system to coincide with the center of mass of the rigid object.

Using the previous expressions, we can now rewrite the total kinetic energy of the rigid object as

$$T = \frac{1}{2}MV^2 + \frac{1}{2} \sum_{i,j} I_{ij} \omega_i \omega_j = T_{CM} + T_{rot} \quad (11.9)$$

The quantity I_{ij} is called the **inertia tensor**, and is a 3 x 3 matrix:

$$\{I\} = \begin{pmatrix} \sum_{\alpha} m_{\alpha} (r_{\alpha,2}^2 + r_{\alpha,3}^2) & -\sum_{\alpha} m_{\alpha} r_{\alpha,1} r_{\alpha,2} & -\sum_{\alpha} m_{\alpha} r_{\alpha,1} r_{\alpha,3} \\ -\sum_{\alpha} m_{\alpha} r_{\alpha,2} r_{\alpha,1} & \sum_{\alpha} m_{\alpha} (r_{\alpha,1}^2 + r_{\alpha,3}^2) & -\sum_{\alpha} m_{\alpha} r_{\alpha,2} r_{\alpha,3} \\ -\sum_{\alpha} m_{\alpha} r_{\alpha,3} r_{\alpha,1} & -\sum_{\alpha} m_{\alpha} r_{\alpha,3} r_{\alpha,2} & \sum_{\alpha} m_{\alpha} (r_{\alpha,1}^2 + r_{\alpha,2}^2) \end{pmatrix} \quad (11.10)$$

Based on the definition of the inertia tensor, we make the following observations:

1. The tensor is symmetric: $I_{ij} = I_{ji}$. Of the 9 parameters, only 6 are free parameters.
2. The non-diagonal tensor elements are called **products of inertia**.
3. The diagonal tensor elements are the moments of inertia with respect to the three coordinate axes of the rotating frame.

11.2 Angular Momentum

The total angular momentum L of the rotating rigid object is equal to the vector sum of the angular momenta of each component of the rigid object. The i^{th} component of L is equal to

$$\begin{aligned} L_i &= \sum_{\alpha} (\vec{r}_{\alpha} \times \vec{p}_{\alpha})_i = \sum_{\alpha} (\vec{r}_{\alpha} \times m_{\alpha} (\vec{\omega} \times \vec{r}_{\alpha}))_i = \sum_{\alpha} m_{\alpha} (\vec{r}_{\alpha} \times \vec{\omega} \times \vec{r}_{\alpha})_i = \sum_{\alpha} m_{\alpha} (r_{\alpha}^2 \vec{\omega} - \vec{r}_{\alpha} (\vec{r}_{\alpha} \cdot \vec{\omega}))_i \\ &= \sum_{\alpha} m_{\alpha} \{ r_{\alpha}^2 \omega_i - r_{\alpha,i} \sum_j (r_{\alpha,j} \omega_j) \} = \sum_j \omega_j \sum_{\alpha} m_{\alpha} \{ r_{\alpha}^2 \delta_{ij} - r_{\alpha,i} r_{\alpha,j} \} = \sum_j I_{ij} \omega_j \end{aligned} \quad (11.11)$$

This equation clearly shows that the angular momentum is in general not parallel to the angular velocity. An example of a system where the angular momentum is directed in a direction different from the direction of the angular velocity is shown in Fig. 11.1.

The rotational kinetic energy can also be rewritten in terms of the angular momentum:

$$T_{rot} = \frac{1}{2} \sum_{i,j} I_{ij} \omega_i \omega_j = \frac{1}{2} \sum_i \omega_i \left(\sum_j I_{ij} \omega_j \right) = \frac{1}{2} \sum_i \omega_i L_i = \frac{1}{2} (\vec{\omega} \cdot \vec{L}) \quad (11.12)$$

11.3 Principal Axes

We always have the freedom to choose our coordinate axes such that the problem we are trying solve is simplified. When we are working on problems that involve the use of the inertia tensor, we can obtain a significant simplification if we can choose our coordinate axes such that the non-diagonal elements are 0. In this case, the inertia tensor would be equal to

$$\{I\} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \quad (11.13)$$

For this inertia tensor we get the following relation between the angular momentum and the angular velocity:

$$L_i = I_i \omega_i \quad (11.14)$$

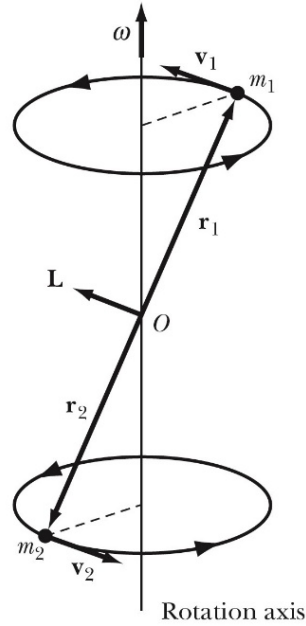


Figure 11.1: A rotating dumbbell is an example of a system in which the angular velocity is not parallel to the angular momentum.

The rotational kinetic energy is equal to

$$T_{rot} = \frac{1}{2} \sum_i I_i \omega_i^2 \quad (11.15)$$

The axes for which the non-diagonal matrix elements vanish are called the **principal axes of inertia**.

The biggest problem we are facing is how do we determine the proper coordinate axes? If the angular velocity vector is directed along one of the three coordinate axes that would get rid of the non-diagonal inertia tensor elements, and we expect to see the following relation between the angular velocity vector and the angular momentum:

$$\vec{L} = I \vec{\omega} \quad (11.16)$$

Substituting the general form of the inertia tensor into this expression, we must require that

$$\begin{aligned} L_1 = I \omega_1 &= I_{11} \omega_1 + I_{12} \omega_2 + I_{13} \omega_3 \\ L_2 = I \omega_2 &= I_{21} \omega_1 + I_{22} \omega_2 + I_{23} \omega_3 \\ L_3 = I \omega_3 &= I_{31} \omega_1 + I_{32} \omega_2 + I_{33} \omega_3 \end{aligned} \quad (11.17)$$

This set of equations can be rewritten as

$$\begin{aligned} (I_{11} - I) \omega_1 + I_{12} \omega_2 + I_{13} \omega_3 &= 0 \\ I_{21} \omega_1 + (I_{22} - I) \omega_2 + I_{23} \omega_3 &= 0 \\ I_{31} \omega_1 + I_{32} \omega_2 + (I_{33} - I) \omega_3 &= 0 \end{aligned} \quad (11.18)$$

This set of equations only has a non-trivial solution if the determinant of the coefficients vanish. This requires that

$$\begin{vmatrix} I_{11} - I & I_{12} & I_{13} \\ I_{21} & I_{22} - I & I_{23} \\ I_{31} & I_{32} & I_{33} - I \end{vmatrix} = 0 \quad (11.19)$$

This requirement leads to three possible values of I . Each of these corresponds to the moment of inertia about one of the principal axes.

11.3.1 Example: Problem 11.13

A three-particle system consists of masses m_i and coordinates (x_1, x_2, x_3) as follows:

$$\begin{aligned} m_1 &= 3m & (b, 0, b) \\ m_2 &= 4m & (b, b, -b) \\ m_3 &= 2m & (-b, b, 0) \end{aligned} \quad (11.20)$$

Find the inertia tensor, the principal axes, and the principal moments of inertia.

We get the elements of the inertia tensor from Eq. 11.13a:

$$\begin{aligned} I_{11} &= \sum_{\alpha} m_{\alpha} (x_{\alpha,2}^2 + x_{\alpha,3}^2) = \\ &= 3m(b^2) + 4m(2b^2) + 2m(b^2) = 13mb^2 \end{aligned} \quad (11.21)$$

Likewise $I_{22} = 16mb^2$ and $I_{33} = 15mb^2$.

$$\begin{aligned} I_{12} &= I_{21} = -\sum_{\alpha} m_{\alpha} x_{\alpha,1} x_{\alpha,2} \\ &= -4m(b^2) - 2m(-b^2) = -2mb^2 \end{aligned} \quad (11.22)$$

Likewise $I_{13} = I_{31} = mb^2$ and $I_{23} = I_{32} = 4mb^2$. Thus, the inertia tensor is

$$\{I\} = mb^2 \begin{bmatrix} 13 & -2 & 1 \\ -2 & 16 & 4 \\ 1 & 4 & 15 \end{bmatrix} \quad (11.23)$$

The principal moments of inertia can be determined by solving the following equation

$$mb^2 \begin{bmatrix} 13 - \lambda & -2 & 1 \\ -2 & 16 - \lambda & 4 \\ 1 & 4 & 15 - \lambda \end{bmatrix} = 0 \quad (11.24)$$

Expanding the determinant gives a cubic equation in λ :

$$\lambda^3 - 44\lambda^2 + 622\lambda - 2820 = 0 \quad (11.25)$$

Solving this equation numerically we obtain the following values for λ

$$\begin{aligned} \lambda_1 &= 10.00 \\ \lambda_2 &= 14.35 \\ \lambda_3 &= 19.65 \end{aligned} \quad (11.26)$$

The principle moments of inertia are equal to

$$\begin{aligned} I_1 &= 10mb^2 \\ I_2 &= 14.35mb^2 \\ I_3 &= 19.65mb^2 \end{aligned} \quad (11.27)$$

To find the principal axes, we substitute the different values of λ into Eq. 11.18 (see examples 11.3 and 11.5 in the textbook):

$$\begin{aligned} (13 - \lambda_i)\omega_{1i} - 2\omega_{2i} + \omega_{3i} &= 0 \\ -2\omega_{1i} + (16 - \lambda_i)\omega_{2i} + 4\omega_{3i} &= 0 \\ \omega_{1i} + 4\omega_{2i} + (15 - \lambda_i)\omega_{3i} &= 0 \end{aligned} \quad (11.28)$$

For $i = 1$, we have $\lambda_1 = 10$:

$$\begin{aligned} 3\omega_{11} - 2\omega_{21} + \omega_{31} &= 0 \\ -2\omega_{11} + 6\omega_{21} + 4\omega_{31} &= 0 \\ \omega_{11} + 4\omega_{21} + 5\omega_{31} &= 0 \end{aligned} \quad (11.29)$$

Solving the first equation for ω_{31} and substituting into the second equation shows that

$$\omega_{11} = \omega_{21} \quad (11.30)$$

Substituting this relation into the third equation shows that

$$\omega_{31} = -\omega_{21} \quad (11.31)$$

or

$$\omega_{11} : \omega_{21} : \omega_{31} = 1 : 1 : -1 \quad (11.32)$$

So, the principal axis associated with I_1 is

$$\frac{1}{\sqrt{3}}(\hat{x} + \hat{y} - \hat{z}) \quad (11.33)$$

Proceeding in the same way gives the other two principal axes:

$$\begin{aligned} i = 2: &-.81\hat{x} + .29\hat{y} - .52\hat{z} \\ i = 3: &-.14\hat{x} + .77\hat{y} + .63\hat{z} \end{aligned} \quad (11.34)$$

We note that the principal axes are mutually orthogonal, as they must be.

Our observation in problem 11.13 that the principal vectors are orthogonal is true in general. We can prove this in the following manner. For the m^{th} principal moment the following relations must hold:

$$L_{im} = I_m \omega_{im} \quad (11.35)$$

$$L_{im} = \sum_k I_{ik} \omega_{km} \quad (11.36)$$

Combining these two equations we obtain

$$\sum_k I_{ik} \omega_{km} = I_m \omega_{im} \quad (11.37)$$

Now multiply both sides of this equation by ω_{in} and sum over i :

$$\sum_{i,k} I_{ik} \omega_{km} \omega_{in} = \sum_i I_m \omega_{im} \omega_{in} \quad (11.38)$$

A similar relation can be obtained for the n^{th} principal moment, multiplied by ω_{km} and summed over k :

$$\sum_{i,k} I_{ki} \omega_{in} \omega_{km} = \sum_k I_n \omega_{km} \omega_{kn} \quad (11.39)$$

If we subtract Eq. 11.39 from Eq. 11.40 we obtain the following result:

$$\begin{aligned} \sum_{i,k} I_{ik} \omega_{km} \omega_{in} - \sum_{i,k} I_{ki} \omega_{in} \omega_{km} &= \sum_{i,k} (I_{ik} - I_{ki}) \omega_{in} \omega_{km} = 0 = \\ &= \sum_i I_m \omega_{im} \omega_{in} - \sum_i I_n \omega_{im} \omega_{in} = (I_m - I_n) \sum_i \omega_{im} \omega_{in} \end{aligned} \quad (11.40)$$

Assuming that the principal momenta are distinct, the previous equation can only be correct if

$$\sum_i \omega_{im} \omega_{in} = \vec{\omega}_m \cdot \vec{\omega}_n = 0 \quad (11.41)$$

which shows the principal axes are orthogonal.

11.4 Transformations of the Inertia Tensor

In our discussion so far we have assumed that the origin of the rotating reference frame coincides with the center of mass of the rigid object. In this Section we will examine what will change if we do not make this assumption.

Consider the two coordinate systems shown in Fig. 11.2. One reference frame, the x frame, has its origin O coincide with the center of mass of the rigid object; the second reference frame, the X frame, has an origin Q that is displaced with respect to the center of mass of the rigid object. The inertia tensor J_{ij} in reference frame X is defined in the same way as it was defined previously:

$$J_{ij} = \sum_{\alpha} m_{\alpha} \left(\delta_{ij} \sum_k X_{\alpha,k}^2 - X_{\alpha,i} X_{\alpha,j} \right) \quad (11.42)$$

The coordinates in the X frame are related to the coordinates in the x frame in the following way:

$$X_i = a_i + x_i \quad (11.43)$$

Using this relation, we can express the inertia tensor in reference frame X in terms of the coordinates in reference frame x :

$$\begin{aligned} J_{ij} &= \sum_{\alpha} m_{\alpha} \left(\delta_{ij} \sum_k (a_k + x_{\alpha,k})^2 - (a_i + x_{\alpha,i})(a_j + x_{\alpha,j}) \right) = \\ &= \sum_{\alpha} m_{\alpha} \left(\delta_{ij} \sum_k x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j} \right) + \sum_{\alpha} m_{\alpha} \left(\delta_{ij} \sum_k (a_k^2 + 2a_k x_{\alpha,k}) - (a_i a_j + a_i x_{\alpha,j} + a_j x_{\alpha,i}) \right) = \\ &= I_{ij} + \sum_{\alpha} m_{\alpha} \left(\delta_{ij} \sum_k a_k^2 - a_i a_j \right) + \sum_{\alpha} m_{\alpha} \left(2\delta_{ij} \sum_k (a_k x_{\alpha,k}) - a_i x_{\alpha,j} - a_j x_{\alpha,i} \right) \end{aligned} \quad (11.44)$$

The last term on the right-hand side is equal to 0 since the origin of the coordinate system x coincides with the center of mass of the object:

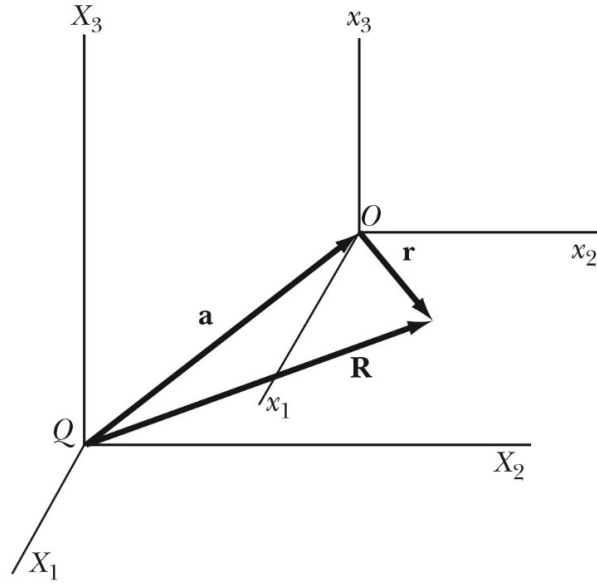


Figure 11.2: Two coordinate systems used to describe our rigid body.

$$\sum_{\alpha} m_{\alpha} x_{\alpha,k} = M r_{cm,k} = 0 \quad (11.45)$$

The relation between the inertia tensor in reference frame X and the inertia tensor in reference frame x is thus given by

$$J_{ij} = I_{ij} + \sum_{\alpha} m_{\alpha} \left(\delta_{ij} \sum_k a_k^2 - a_i a_j \right) = I_{ij} + M (\delta_{ij} a^2 - a_i a_j) \quad (11.46)$$

This relation is called the **Steiner's parallel-axis theorem** and is one example of how coordinate transformations affect the inertia tensor.

The transformation discussed so far is a simple translation. Other important transformations are rotations. In Chapter 1 we discussed many examples of rotations, and determined that the most general way to express rotations is by using the rotation matrix λ :

$$x_i = \sum_j \lambda_{ji} x'_j \quad (11.47)$$

Since this transformation rule is valid for vectors in general, the same rule can be used to describe the transformation of the angular momentum and angular velocity vectors:

$$L_i = \sum_j \lambda_{ji} L'_j \quad (11.48)$$

$$\omega_i = \sum_j \lambda_{ji} \omega'_j \quad (11.49)$$

In order to determine the relation between the inertia tensor in the two coordinate frames, we use the fact that the angular momentum is the product of the inertia tensor and the angular velocity, in both frames:

$$L_k = \sum_l I_{kl} \omega_l \quad (11.50)$$

and

$$L'_k = \sum_l I'_{kl} \omega'_l \quad (11.51)$$

In order to relate the inertia tensors, we use the coordinate transformations for L and ω :

$$\sum_j \lambda_{jk} L'_j = \sum_l I_{kl} \sum_m \lambda_{ml} \omega'_m \quad (11.52)$$

This equation can be simplified if we multiply each side by λ_k and sum over k :

$$\begin{aligned} \sum_k \lambda_{ik} \left(\sum_j \lambda_{jk} L'_j \right) &= \sum_j \left\{ \sum_k \left(\lambda_{jk} \lambda_{ik} \right) L'_j \right\} = \sum_j \left\{ \delta_{ji} L'_j \right\} = L'_i = \\ &= \sum_k \lambda_{ik} \left(\sum_l I_{kl} \sum_m \lambda_{ml} \omega'_m \right) = \sum_m \left\{ \sum_{k,l} \lambda_{ik} \lambda_{ml} I_{kl} \right\} \omega'_m \end{aligned} \quad (11.53)$$

where we have used the orthogonal properties of the rotation matrix. Using the relation between the angular momentum and the angular velocity in the rotated coordinate frame we see that the inertia tensors in the two coordinate frames are related as follows:

$$I'_{im} = \sum_{k,l} \lambda_{ik} \lambda_{ml} I_{kl} = \sum_{k,l} \lambda_{ik} I_{kl} \lambda^t_{lm} \quad (11.54)$$

where λ^t is the transposed matrix. In tensor notation we can rewrite this relation as

$$\{I'\} = \{\lambda\} \{I\} \{\lambda^t\} \quad (11.55)$$

It turns out that for any inertia tensor we can find a rotation such that the inertia tensor in the rotated frame is a diagonal matrix (all non-diagonal elements are equal to 0).

We thus have seen two different approaches to diagonalize the inertia tensor: 1) find the principal axes of inertia, and 2) find the proper rotation matrix.

11.4.1 Example: Problem 11.16

Consider the following inertia tensor:

$$\{I\} = \left\{ \begin{array}{ccc} \frac{1}{2}(A+B) & \frac{1}{2}(A-B) & 0 \\ \frac{1}{2}(A-B) & \frac{1}{2}(A+B) & 0 \\ 0 & 0 & C \end{array} \right\} \quad (11.56)$$

Perform a rotation of the coordinate system by an angle θ about the x_3 axis. Evaluate the transformed tensor elements, and show that the choice $\theta = \pi/4$ renders the inertia tensor diagonal with elements A , B , and C .

The rotation matrix is

$$\{\lambda\} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (11.57)$$

The moment of inertia tensor transforms according to

$$\{I'\} = \{\lambda\} \{I\} \{\lambda'\} \quad (11.58)$$

That is

$$\begin{aligned}
\{I'\} &= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}(A+B) & \frac{1}{2}(A-B) & 0 \\ \frac{1}{2}(A-B) & \frac{1}{2}(A+B) & 0 \\ 0 & 0 & C \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \\
&= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}(A+B)\cos\theta + \frac{1}{2}(A-B)\sin\theta & -\frac{1}{2}(A+B)\sin\theta + \frac{1}{2}(A-B)\cos\theta & 0 \\ \frac{1}{2}(A-B)\cos\theta + \frac{1}{2}(A+B)\sin\theta & -\frac{1}{2}(A-B)\sin\theta + \frac{1}{2}(A+B)\cos\theta & 0 \\ 0 & 0 & C \end{bmatrix} = \\
&= \begin{bmatrix} \frac{1}{2}(A+B)\cos^2\theta + (A-B)\cos\theta\sin\theta + \frac{1}{2}(A+B)\sin^2\theta & & \\ -\frac{1}{2}(A-B)\sin^2\theta + \frac{1}{2}(A-B)\cos^2\theta & & \\ 0 & & \\ \frac{1}{2}(A-B)\cos^2\theta - \frac{1}{2}(A-B)\sin^2\theta & 0 & \\ \frac{1}{2}(A+B)\sin^2\theta - (A-B)\sin\theta\cos\theta + \frac{1}{2}(A+B)\cos^2\theta & 0 & \\ 0 & 0 & C \end{bmatrix} \tag{11.59}
\end{aligned}$$

or

$$\{I'\} = \begin{bmatrix} \frac{1}{2}(A+B) + (A-B)\cos\theta\sin\theta & \frac{1}{2}(A-B)\cos^2\theta - \frac{1}{2}(A-B)\sin^2\theta & 0 \\ -\frac{1}{2}(A-B)\sin^2\theta + \frac{1}{2}(A-B)\cos^2\theta & \frac{1}{2}(A+B) - (A-B)\cos\theta\sin\theta & 0 \\ 0 & 0 & C \end{bmatrix} \tag{11.60}$$

If $\theta = \pi/4$, $\sin\theta = \cos\theta = 1/\sqrt{2}$. Then,

$$\boxed{\{I'\} = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}} \tag{11.61}$$

11.5 Euler Angles

Any rotation between different coordinate systems can be expressed in terms of three successive rotations around the coordinate axes. When we consider the transformation from the fixed coordinate system x' to the body coordinate system x , we call the three angles the Euler angles ϕ , θ , and ψ (see Fig. 11.3). The total transformation matrix is the product of the individual transformations (note the order of the transformations):

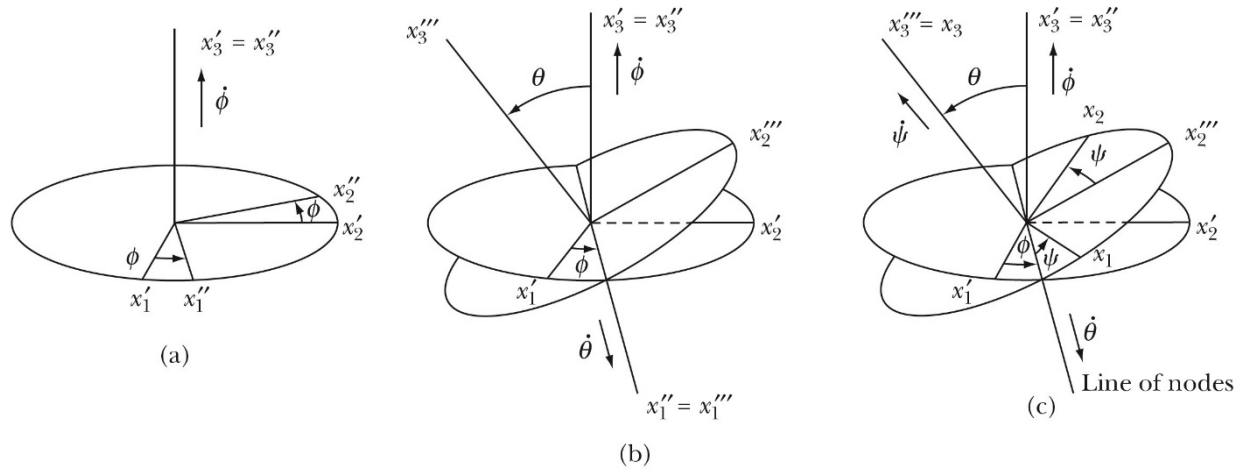


Figure 11.3: The Euler angles used to transform the fixed coordinate system x' into the body coordinate system x .

$$\lambda = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \quad (11.62)$$

$$= \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

With each of the three rotations we can associate an angular velocity ω . To express the angular velocity in the body coordinate system, we can use Fig. 11.3c.

1. ω_ϕ : Figure 3c shows that the angular velocity ω_ϕ is directed in the $x_2''' - x_3'''$ plane. Its projection along the x_3''' axis, which is also the x_3 axis, is equal to

$$\dot{\phi}_3 = \dot{\phi}\cos\theta \quad (11.63)$$

The projection along the x_2''' axis is equal to

$$\dot{\phi}_{2'''} = \dot{\phi}\sin\theta \quad (11.64)$$

Fig. 11.3c shows that when we project this projection along the x_1 and x_2 axes we obtain the following components in the body coordinate system:

$$\dot{\phi}_1 = \dot{\phi}_{2'''}\sin\psi = \dot{\phi}\sin\theta\sin\psi \quad (11.65)$$

$$\dot{\phi}_2 = \dot{\phi}_{2'''}\cos\psi = \dot{\phi}\sin\theta\cos\psi \quad (11.66)$$

2. ω_θ : Fig. 11.3c shows that the angular velocity ω_θ is directed in the $x_1''' - x_2'''$ plane. Its projection along the x_3''' axis, which is also the x_3 axis, is equal to 0.

$$\dot{\theta}_3 = 0 \quad (11.67)$$

Fig. 11.3c shows that when we project ω_θ along the x_1 and x_2 axes we obtain the following components in the body coordinate system:

$$\dot{\theta}_1 = \dot{\theta} \cos \psi \quad (11.68)$$

$$\dot{\theta}_2 = -\dot{\theta} \sin \psi \quad (11.69)$$

3. ω_ψ : Fig. 11.3c shows that the angular velocity ω_ψ is directed along the x_3''' axis, which is also the x_3 axis. The components along the other body axes are 0. Thus:

$$\dot{\psi}_1 = 0 \quad (11.70)$$

$$\dot{\psi}_2 = 0 \quad (11.71)$$

$$\dot{\psi}_3 = \dot{\psi} \quad (11.72)$$

The angular velocity, in the body frame, is thus equal to

$$\vec{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} \dot{\phi}_1 + \dot{\theta}_1 + \dot{\psi}_1 \\ \dot{\phi}_2 + \dot{\theta}_2 + \dot{\psi}_2 \\ \dot{\phi}_3 + \dot{\theta}_3 + \dot{\psi}_3 \end{pmatrix} = \begin{pmatrix} \dot{\phi} \sin \theta \sin \psi + \dot{\theta} \cos \psi \\ \dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi \\ \dot{\phi} \cos \theta + \dot{\psi} \end{pmatrix} \quad (11.73)$$

11.6 The Force-Free Euler Equations

Let us assume for the moment that the coordinate axes correspond to the principal axes of the body. In that case, we can write the kinetic energy of the body in the following manner:

$$T = \frac{1}{2} \sum_i I_i \omega_i^2 \quad (11.74)$$

where I_i are the principal moments of the rigid body. If for now we consider that the rigid object is carrying out a force-free motion ($U = 0$) then the Lagrangian L will be equal to the kinetic energy T . The motion of the object can be described in terms of the Euler angles, which can serve as the generalized coordinates of the motion. Consider the three equations of motion for the three generalized coordinates:

1. The Euler angle ϕ : Lagrange's equation for the coordinate ϕ is

$$0 = \frac{\partial L}{\partial \phi} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} = \frac{\partial T}{\partial \phi} - \frac{d}{dt} \frac{\partial T}{\partial \dot{\phi}} = \sum_i \frac{\partial T}{\partial \omega_i} \frac{\partial \omega_i}{\partial \phi} - \frac{d}{dt} \sum_i \frac{\partial T}{\partial \omega_i} \frac{\partial \omega_i}{\partial \dot{\phi}} = \sum_i I_i \omega_i \frac{\partial \omega_i}{\partial \phi} - \frac{d}{dt} \sum_i I_i \omega_i \frac{\partial \omega_i}{\partial \dot{\phi}} = 0 \quad (11.75)$$

Differentiating the angular velocity with respect to the coordinate ϕ we find

$$\begin{aligned}
\frac{\partial \omega_1}{\partial \phi} &= 0 & \frac{\partial \omega_1}{\partial \phi} &= \sin\theta \sin\psi \\
\frac{\partial \omega_2}{\partial \phi} &= 0 & \frac{\partial \omega_2}{\partial \phi} &= \sin\theta \cos\psi \\
\frac{\partial \omega_3}{\partial \phi} &= 0 & \frac{\partial \omega_3}{\partial \phi} &= \cos\theta
\end{aligned} \tag{11.76}$$

and Lagrange's equation of motion becomes

$$\frac{d}{dt} \{I_1 \omega_1 \sin\theta \sin\psi + I_2 \omega_2 \sin\theta \cos\psi + I_3 \omega_3 \cos\theta\} = 0 \tag{11.77}$$

2. The Euler angle θ : Lagrange's equation for the coordinate θ is

$$0 = \frac{\partial T}{\partial \theta} - \frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}} = \sum_i \frac{\partial T}{\partial \omega_i} \frac{\partial \omega_i}{\partial \theta} - \frac{d}{dt} \sum_i \frac{\partial T}{\partial \omega_i} \frac{\partial \omega_i}{\partial \dot{\theta}} = \sum_i I_i \omega_i \frac{\partial \omega_i}{\partial \theta} - \frac{d}{dt} \sum_i I_i \omega_i \frac{\partial \omega_i}{\partial \dot{\theta}} = 0 \tag{11.78}$$

Differentiating the angular velocity with respect to the coordinate θ we find

$$\begin{aligned}
\frac{\partial \omega_1}{\partial \theta} &= \dot{\phi} \cos\theta \sin\psi & \frac{\partial \omega_1}{\partial \dot{\theta}} &= \cos\psi \\
\frac{\partial \omega_2}{\partial \theta} &= \dot{\phi} \cos\theta \cos\psi & \frac{\partial \omega_2}{\partial \dot{\theta}} &= -\sin\psi \\
\frac{\partial \omega_3}{\partial \theta} &= -\dot{\phi} \sin\theta & \frac{\partial \omega_3}{\partial \dot{\theta}} &= 0
\end{aligned} \tag{11.79}$$

and Lagrange's equation of motion becomes

$$\dot{\phi} \{ (I_1 \omega_1 \sin\psi + I_2 \omega_2 \cos\psi) \cos\theta - I_3 \omega_3 \sin\theta \} - \frac{d}{dt} \{ I_1 \omega_1 \cos\psi - I_2 \omega_2 \sin\psi \} = 0 \tag{11.80}$$

3. The Euler angle ψ : Lagrange's equation for the coordinate ψ is

$$0 = \frac{\partial T}{\partial \psi} - \frac{d}{dt} \frac{\partial T}{\partial \dot{\psi}} = \sum_i \frac{\partial T}{\partial \omega_i} \frac{\partial \omega_i}{\partial \psi} - \frac{d}{dt} \sum_i \frac{\partial T}{\partial \omega_i} \frac{\partial \omega_i}{\partial \dot{\psi}} = \sum_i I_i \omega_i \frac{\partial \omega_i}{\partial \psi} - \frac{d}{dt} \sum_i I_i \omega_i \frac{\partial \omega_i}{\partial \dot{\psi}} = 0 \tag{11.81}$$

Differentiating the angular velocity with respect to the coordinate ψ we find

$$\begin{aligned}
\frac{\partial \omega_1}{\partial \psi} &= \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi = \omega_2 & \frac{\partial \omega_1}{\partial \dot{\psi}} &= 0 \\
\frac{\partial \omega_2}{\partial \psi} &= -\dot{\phi} \sin\theta \sin\psi - \dot{\theta} \cos\psi = -\omega_1 & \frac{\partial \omega_2}{\partial \dot{\psi}} &= 0 \\
\frac{\partial \omega_3}{\partial \psi} &= 0 & \frac{\partial \omega_3}{\partial \dot{\psi}} &= 1
\end{aligned} \tag{11.82}$$

and Lagrange's equation of motion becomes

$$I_1 \omega_1 \omega_2 - I_2 \omega_2 \omega_1 - \frac{d}{dt} \{ I_3 \omega_3 \} = (I_1 - I_2) \omega_1 \omega_2 - \frac{d}{dt} \{ I_3 \omega_3 \} = (I_1 - I_2) \omega_1 \omega_2 - I_3 \dot{\omega}_3 = 0 \tag{11.83}$$

Of all three equations of motion, the last one is the only one to contain just the components of the angular velocity. Since our choice of the x_3 axis was arbitrary, we expect that similar relations should exist for the other two axes. The set of three equations we obtain in this way are called the **Euler equations**:

$$(I_1 - I_2)\omega_1\omega_2 - I_3\dot{\omega}_3 = 0 \quad (11.84)$$

$$(I_2 - I_3)\omega_2\omega_3 - I_1\dot{\omega}_1 = 0 \quad (11.85)$$

$$(I_3 - I_1)\omega_3\omega_1 - I_2\dot{\omega}_2 = 0 \quad (11.86)$$

As an example of how we use Euler's equations, consider a symmetric top. The top will have two different principal moments: $I_1 = I_2$ and I_3 . In this case, the first Euler equation reduces to

$$I_3\dot{\omega}_3 = 0 \quad (11.87)$$

or

$$\omega_3(t) = \text{constant} = \omega_3 \quad (11.88)$$

The other two Euler equations can be rewritten as

$$\dot{\omega}_1 = -\left(\frac{I_3 - I_1}{I_1}\omega_3\right)\omega_2 = -\omega\omega_2 \quad (11.89)$$

$$\dot{\omega}_2 = \left(\frac{I_3 - I_1}{I_1}\omega_3\right)\omega_1 = \omega\omega_1 \quad (11.90)$$

where ω is

$$\omega = \left(\frac{I_3 - I_1}{I_1}\omega_3\right) \quad (11.91)$$

This set of equations has the following solution:

$$\omega_1(t) = A\cos\omega t \quad (11.92)$$

$$\omega_2(t) = A\sin\omega t \quad (11.93)$$

The magnitude of the angular velocity of the system is constant since

$$|\vec{\omega}| = \sqrt{(\omega_1^2(t) + \omega_2^2(t) + \omega_3^2)} = \sqrt{A^2 + \omega_3^2} \quad (11.94)$$

The angular velocity vector traces out a cone in the body frame (it precesses around the x_3 axis - see Fig. 11.4). The rate with which the angular velocity vector precesses around the x_3 axis is determined by the value of ω . When the principal moment I_3 and the principal moment I_1 are similar, ω becomes very small.

Since we have assumed that there are no external forces and torques acting on the system, the angular momentum of the system will be constant in the fixed reference frame. If the angular momentum is initially pointing along the x'_3 axis it will continue to point along this axis (see Fig. 11.5). Since there are no external forces and torques acting on the system, the rotation kinetic energy of the system must be constant. Thus

$$T_{rot} = \frac{1}{2}\vec{\omega} \cdot \vec{L} \quad (11.95)$$

Since the angle between the angular velocity vector and the angular momentum vector must be constant, the angular velocity vector must trace out a space cone around the x'_3 (see Fig. 11.5).

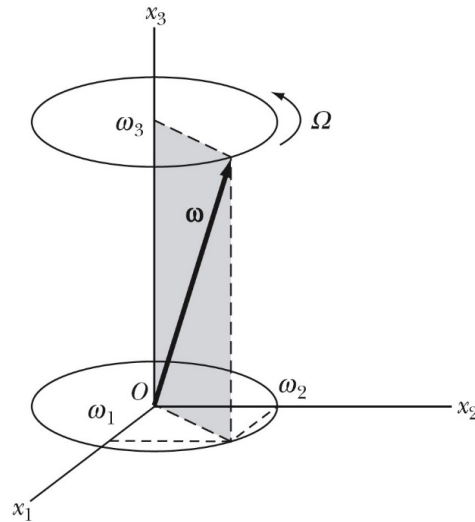


Figure 11.4: The angular velocity of a force-free symmetric top, precessing around the x_3 axis in the body frame.

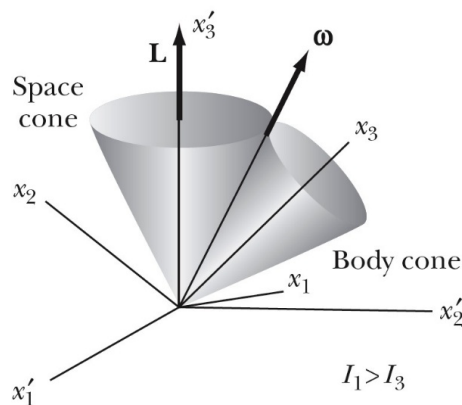


Figure 11.5: The angular velocity of a force-free symmetric top, tracing out a space-cone around the x'_3 axis in the body frame.

11.6.1 Example: Problem 11.27

A symmetric body moves without the influence of forces or torques. Let x_3 be the symmetry axis of the body and L be along x'_3 . The angle between the angular velocity vector and x_3 is α . Let ω and L initially be in the x_2 - x_3 plane. What is the angular velocity of the symmetry axis about L in terms of I_1 , I_3 , ω , and α ?

The coordinate system we will use to solve this problem is shown in Fig. 11.6. The initial conditions are given by

$$L_1 = 0 = I_1 \omega_1 \quad (11.96)$$

$$L_2 = L \sin \theta = I_1 \omega_2 = I_1 \omega \sin \alpha \quad (11.97)$$

$$L_3 = L \cos \theta = I_3 \omega_3 = I_3 \omega \cos \alpha \quad (11.98)$$

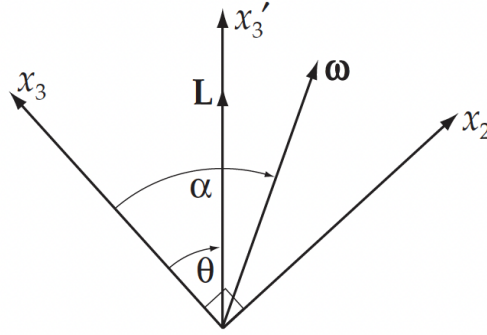


Figure 11.6: Coordinate system used to solve Problem 11.27.

Thus

$$\tan\theta = \frac{L_2}{L_3} = \frac{I_1}{I_3} \tan\alpha \quad (11.99)$$

Use Eq. (11.102) in the textbook:

$$\omega_3 = \dot{\phi} \cos\theta + \dot{\psi} \quad (11.100)$$

Since $\omega_3 = \omega \cos\alpha$, we have

$$\dot{\phi} \cos\theta = \omega \cos\alpha - \dot{\psi} \quad (11.101)$$

Using Eq. (11.131) in the textbook

$$\dot{\psi} = -\Omega = -\frac{I_3 - I_1}{I_1} \omega_3 \quad (11.102)$$

Eq. (11.101) becomes

$$\dot{\phi} \cos\theta = \frac{I_3}{I_1} \omega \cos\alpha \quad (11.103)$$

We can use Eq. (11.99) to express $\cos\theta$ in terms of $\tan\alpha$

$$\cos\theta = \frac{I_3}{[I_3^2 + I_1^2 \tan^2\alpha]^{1/2}} \quad (11.104)$$

Substituting this expression for $\cos\theta$ into (11.103) gives

$$\dot{\phi} = \frac{\omega}{I_1} \sqrt{I_1^2 \sin^2\alpha + I_3^2 \cos^2\alpha} \quad (11.105)$$

Equation 11.99 shows that

1. if $I_1 > I_3$ (prolate body), $\theta > \alpha$
2. if $I_1 < I_3$ (oblate body), $\theta < \alpha$

The space and body cones for these two different scenarios are shown in Fig. 11.7.

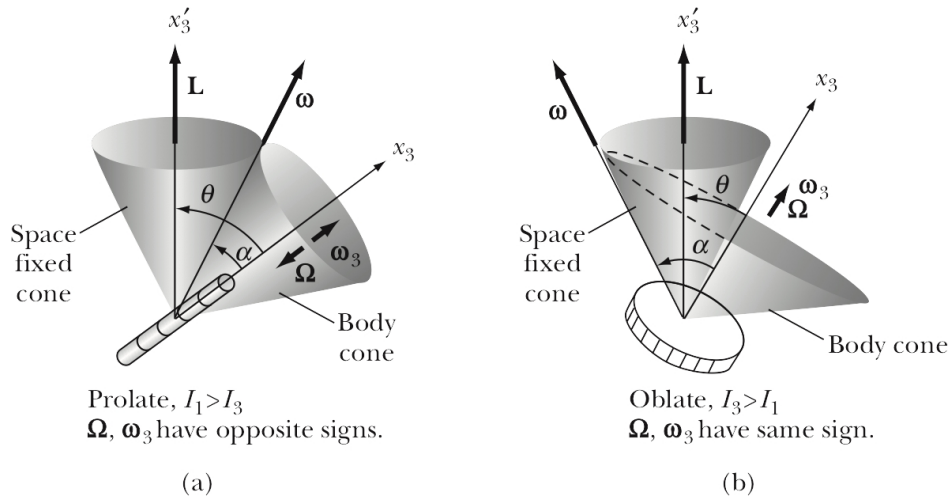


Figure 11.7: Space and body cones for prolate and oblate bodies

11.7 The Euler Equations in a Force Field

When the external forces and torques acting on the system are not equal to 0, we can not use the method we have used in the previous section to obtain expressions for the angular velocity and acceleration. The procedure used in the previous section relied on the fact that the potential energy U is 0 in a force-free environment, and therefore, the Lagrangian L is equal to the kinetic energy T .

When the external forces and torques are not equal to 0, the angular momentum of the system is not conserved:

$$\left(\frac{d\vec{L}}{dt} \right)_{\text{fixed}} = \vec{N} \quad (11.106)$$

Note that this relation only holds in the fixed reference frame since this is the only good inertial reference frame. In Chapter 10 we looked at the relation between parameters specified in the fixed reference frame compared to parameters specified in the rotating reference frame, and we can use this relation to correlate the rate of change of the angular momentum vector in the fixed reference frame with the rate of change of the angular momentum vector in the rotating reference frame:

$$\vec{N} = \left(\frac{d\vec{L}}{dt} \right)_{\text{fixed}} = \left(\frac{d\vec{L}}{dt} \right)_{\text{rotating}} + \vec{\omega} \times \vec{L} \quad (11.107)$$

This relation can be used to generate three separate relations by projecting the vectors along the three body axes:

$$N_1 = \frac{dL_1}{dt} + (\vec{\omega} \times \vec{L})_1 = \frac{dL_1}{dt} + (\omega_2 L_3 - \omega_3 L_2) = I_1 \dot{\omega}_1 - (I_2 - I_3) \omega_2 \omega_3 \quad (11.108)$$

$$N_2 = \frac{dL_2}{dt} + (\vec{\omega} \times \vec{L})_2 = \frac{dL_2}{dt} + (\omega_3 L_1 - \omega_1 L_3) = I_2 \dot{\omega}_2 - (I_3 - I_1) \omega_3 \omega_1 \quad (11.109)$$

$$N_3 = \frac{dL_3}{dt} + (\vec{\omega} \times \vec{L})_3 = \frac{dL_3}{dt} + (\omega_1 L_2 - \omega_2 L_1) = I_3 \dot{\omega}_3 - (I_1 - I_2) \omega_1 \omega_2 \quad (11.110)$$

These equations are the Euler equations for the motion of the rigid body in a force field. In the absence of a torque, these equations reduce to the force-free Euler equations.

11.7.1 Example: Motion of a Symmetric Top with One Point Fixed

In order to describe the motion of a top, which has its tip fixed, we use two coordinate systems whose origins coincide (see Fig. 11.8). Since the origins coincide, the transformation between coordinate systems can be described in terms of the Euler angles, and the equations of motion will be the Euler equations:

$$(I_1 - I_2)\omega_1\omega_2 - I_3\dot{\omega}_3 = 0 \quad (11.111)$$

$$(I_2 - I_3)\omega_2\omega_3 - I_1\dot{\omega}_1 = N_1 \quad (11.112)$$

$$(I_3 - I_1)\omega_3\omega_1 - I_2\dot{\omega}_2 = N_2 \quad (11.113)$$

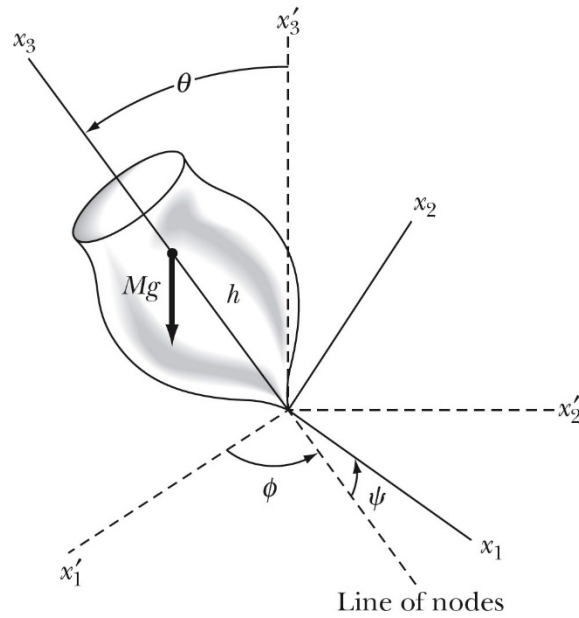


Figure 11.8: Spinning top with fixed tip.

Since the top is symmetric around the x_3 axis, its principal moments of inertia with respect to the x_1 and x_2 axes are identical. The Euler equations now become

$$I_3\dot{\omega}_3 = 0 \quad (11.114)$$

$$(I_1 - I_3)\omega_2\omega_3 - I_1\dot{\omega}_1 = N_1 \quad (11.115)$$

$$-(I_1 - I_3)\omega_3\omega_1 - I_1\dot{\omega}_2 = N_2 \quad (11.116)$$

The first equation immediately tells us that

$$\omega_3 = \text{constant} \quad (11.117)$$

The motion of the top is often described in terms of the motion of its rotating axes. The kinetic energy of the system is equal to

$$T = \frac{1}{2} \sum_i I_i \omega_i^2 = \frac{1}{2} I_1 (\omega_1^2 + \omega_2^2) + \frac{1}{2} I_3 \omega_3^2 = \frac{1}{2} I_1 (\dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2) + \frac{1}{2} I_3 (\dot{\phi} \cos \theta + \dot{\psi})^2 \quad (11.118)$$

The potential energy of the system, assuming the center of mass of the top is located a distance h from the tip, is equal to

$$U = Mgh\cos\theta \quad (11.119)$$

The Lagrangian is thus equal to

$$L = T - U = \frac{1}{2}I_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \frac{1}{2}I_3(\dot{\phi}\cos\theta + \dot{\psi})^2 - Mgh\cos\theta \quad (11.120)$$

The Lagrangian does not depend on ϕ and ψ , and thus

$$\frac{\partial L}{\partial \phi} = 0 = \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} \quad (11.121)$$

$$\frac{\partial L}{\partial \psi} = 0 = \frac{d}{dt} \frac{\partial L}{\partial \dot{\psi}} \quad (11.122)$$

We thus conclude that the angular momenta associated with the Euler angles ϕ and ψ are constant:

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi}(I_1\sin^2\theta + I_3\cos^2\theta) + \dot{\psi}I_3\cos\theta = \text{constant} \quad (11.123)$$

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3(\dot{\psi} + \dot{\phi}\cos\theta) = I_3\omega_3 = \text{constant} \quad (11.124)$$

Expressing the momenta in terms of the Euler angles ϕ and ψ allows us to express the rate of change of these Euler angles in terms of the angular momenta:

$$\dot{\phi} = \frac{p_\phi - p_\psi\cos\theta}{I_1\sin^2\theta} \quad (11.125)$$

Since there are no non-conservative forces acting on the top, the total energy E of the system is conserved. Thus

$$E = T + U = \frac{1}{2}I_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \frac{1}{2}I_3(\dot{\phi}\cos\theta + \dot{\psi})^2 + Mgh\cos\theta = \text{constant} \quad (11.126)$$

The total energy can be rewritten in terms of the angular momenta:

$$\begin{aligned} E &= \frac{1}{2}I_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \frac{1}{2}I_3(\dot{\phi}\cos\theta + \dot{\psi})^2 + Mgh\cos\theta = \\ &= \frac{1}{2}I_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \frac{1}{2}I_3\omega_3^2 + Mgh\cos\theta = \\ &= \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_1\dot{\phi}^2\sin^2\theta + \frac{1}{2}I_3\omega_3^2 + Mgh\cos\theta = \\ &= \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}\frac{(p_\phi - p_\psi\cos\theta)^2}{I_1\sin^2\theta} + \frac{1}{2}I_3\omega_3^2 + Mgh\cos\theta \end{aligned} \quad (11.127)$$

Since the angular velocity with respect to the x_3 axis is constant, we can subtract it from the energy E to get the effective energy E' (note: this is equivalent to choosing the zero point of the energy scale). Thus

$$E' = E - \frac{1}{2}I_3\omega_3^2 = \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}\frac{(p_\phi - p_\psi\cos\theta)^2}{I_1\sin^2\theta} + Mgh\cos\theta = \text{constant} \quad (11.128)$$

The effective energy only depends on the angle θ and on $d\theta/dt$ since the angular momenta are constants. The manipulations we have carried out have reduced the three-dimensional problem to a one-dimensional problem. The first term in the effective energy is the kinetic energy associated with the rotation around the x_1 axis. The last two terms depend only on the angle θ and not on the angular velocity $d\theta/dt$. These terms are what we could call the effective potential energy, defined as

$$V(\theta) = \frac{1}{2} \frac{(p_\phi - p_\psi \cos\theta)^2}{I_1 \sin^2\theta} + Mgh \cos\theta \quad (11.129)$$

The effective potential becomes large when the angle approaches 0 and π . The angular dependence of the effective potential is shown in Fig. 11.9. If the total effective energy of the system is E_1' , we expect the angle θ to vary between θ_1 and θ_2 . We thus expect that the angle of inclination of the top will vary between these two extremes.

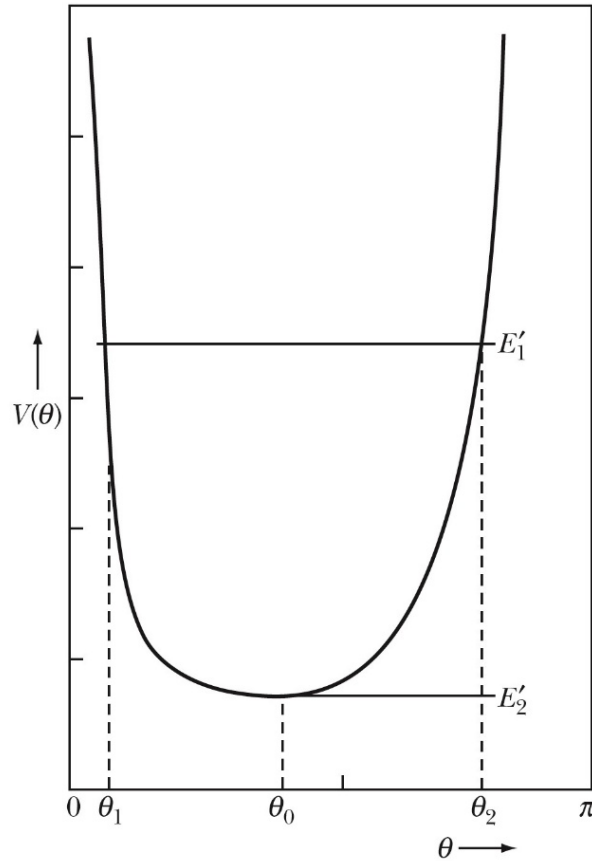


Figure 11.9: The effective potential of a rotating top.

The minimum effective energy that the system can have is E_2' . The corresponding angle can be found by requiring

$$\left. \frac{\partial V}{\partial \theta} \right|_{\theta=\theta_0} = 0 \quad (11.130)$$

This requirement can be rewritten as a quadratic equation of a parameter β , where β is defined as

$$\beta = p_\phi - p_\psi \cos\theta_0 \quad (11.131)$$

In general, there are two solutions to this quadratic equation. Since β is a real number, the solution must be real, and this requires that

$$1 - \frac{4MghI_1 \cos\theta_0}{p_\psi^2} \geq 0 \quad (11.132)$$

This equation can be rewritten as

$$p_\psi^2 = (I_3\omega_3)^2 \geq 4MghI_1\cos\theta_0 \quad (11.133)$$

When we study a spinning top, the spin axis is oriented such that $\theta_0 < \pi/2$. The previous equation can then be rewritten as

$$I_3\omega_3 \geq \sqrt{4MghI_1\cos\theta_0} \quad (11.134)$$

or

$$\omega_3 \geq \frac{2}{I_3} \sqrt{MghI_1\cos\theta_0} \quad (11.135)$$

There is thus a minimum angular velocity the system must have in order to produce stable precession. The rate of precession can be found by calculating

$$\dot{\phi} = \frac{p_\phi - p_\psi\cos\theta_0}{I_1\sin^2\theta_0} = \frac{\beta}{I_1\sin^2\theta_0} \quad (11.136)$$

Since β has two possible values, we expect to see two different precession rates: one resulting in fast precession, and one resulting in slow precession.

When the angle of inclination is not equal to θ_0 , the system will oscillate between two limiting values of θ . The precession rate will be a function of θ and can vary between positive and negative values, depending on the values of the angular momenta. The phenomenon is called nutation, and possible nutation patterns are shown in Fig. 11.10. The type of nutation depends on the initial conditions.

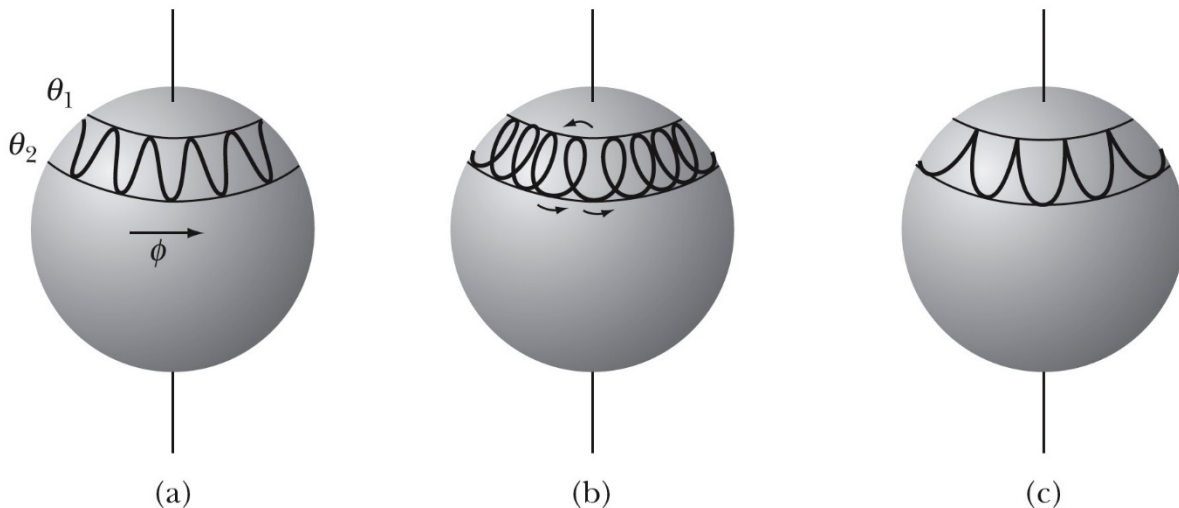


Figure 11.10: The nutation of a rotating top.

11.7.2 Example: Problem 11.30

Investigate the equation for the turning points of the nutational motion by setting $d\theta/dt = 0$ in the equation of the effective energy. Show that the resulting equation is cubic in $\cos\theta$ and has two real roots and one imaginary root.

If we set $\dot{\theta} = 0$ in the equation for the effective energy we obtain

$$E' = V(\theta) = \frac{(P_\phi - P_\psi\cos\theta)^2}{2I_{12}(1 - \cos^2\theta)} + Mgh\cos\theta \quad (11.137)$$

Re-arranging, this equation can be written as

$$(2MghI_{12})\cos^3\theta - (2E'I_{12} + P_\psi^2)\cos^2\theta + 2(P_\phi P_\psi - MghI_{12})\cos\theta + (2E'I_{12} - P_\phi^2) = 0 \quad (11.138)$$

which is cubic in $\cos\theta$. Equation 11.138 has the form shown in Fig. 11.11. Two of the roots occur in the region $-1 < \cos\theta < 1$. One root lies outside this range and is therefore imaginary.

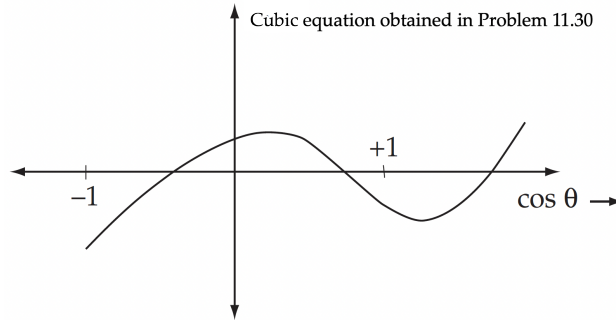


Figure 11.11: Equation 11.138, obtained in Problem 11.30, as function of $\cos\theta$.

11.8 Stability of Rigid-Body Rotations

The rotation of a rigid body is stable if the system, when perturbed from its equilibrium condition, carries out small oscillations about it. Consider we use the principal axes of rotation to describe the motion, and we choose these axes such that $I_3 > I_2 > I_1$. If the system rotates around the x_1 axis we can write the angular velocity vector as

$$\vec{\omega} = \omega_1 \hat{x}_1 \quad (11.139)$$

Consider what happens when we apply a small perturbation around the other two principal axes such that the angular velocity becomes

$$\vec{\omega} = \omega_1 \hat{x}_1 + \lambda \hat{x}_2 + \mu \hat{x}_3 \quad (11.140)$$

The corresponding Euler equations are

$$(I_1 - I_2)\omega_1\omega_2 - I_3\dot{\omega}_3 = (I_1 - I_2)\lambda\omega_1 - I_3\dot{\mu} = 0 \quad (11.141)$$

$$(I_2 - I_3)\omega_2\omega_3 - I_1\dot{\omega}_1 = (I_2 - I_3)\lambda\mu - I_1\dot{\omega}_1 = 0 \quad (11.142)$$

$$(I_3 - I_1)\omega_3\omega_1 - I_2\dot{\omega}_2 = (I_3 - I_1)\mu\omega_1 - I_2\dot{\lambda} = 0 \quad (11.143)$$

Since we are talking about small perturbations from the equilibrium state, $\lambda\mu$ will be small and can be set to 0. The second equation can thus be used to conclude that

$$\omega_1 = \text{constant} \quad (11.144)$$

The remaining equations can be rewritten as

$$\dot{\mu} = \left(\frac{I_1 - I_2}{I_3} \omega_1 \right) \lambda \quad (11.145)$$

$$\dot{\lambda} = \left(\frac{I_3 - I_1}{I_2} \omega_1 \right) \mu \quad (11.146)$$

The last equation can be differentiated to obtain

$$\ddot{\lambda} = \frac{d\dot{\lambda}}{dt} = \left(\frac{I_3 - I_1}{I_2} \omega_1 \right) \frac{d\mu}{dt} = \left(\frac{I_3 - I_1}{I_2} \omega_1 \right) \left(\frac{I_1 - I_2}{I_3} \omega_1 \right) \dot{\lambda} = - \left(\frac{(I_3 - I_1)(I_2 - I_1)}{I_2 I_3} \omega_1^2 \right) \dot{\lambda} \quad (11.147)$$

The term within the parenthesis is positive since we assumed that $I_3 > I_2 > I_1$. This differential equation has the following solution:

$$\lambda(t) = A_\lambda \cos(\Omega_1 t) + B_\lambda \sin(\Omega_1 t) \quad (11.148)$$

where

$$\omega_1 = \Omega_1 \sqrt{\frac{(I_3 - I_1)(I_2 - I_1)}{I_2 I_3}} \quad (11.149)$$

When we look at the perturbation around the x_3 axis we find the following differential equation

$$\ddot{\mu} = \frac{d\dot{\mu}}{dt} = \left(\frac{I_1 - I_2}{I_3} \omega_1 \right) \frac{d\lambda}{dt} = \left(\frac{I_1 - I_2}{I_3} \omega_1 \right) \left(\frac{I_3 - I_1}{I_2} \omega_1 \right) \dot{\mu} = - \left(\frac{(I_2 - I_1)(I_3 - I_1)}{I_2 I_3} \omega_1^2 \right) \dot{\mu} \quad (11.150)$$

The solution of the second-order differential equation is

$$\mu(t) = A_\mu \cos(\Omega_1 t) + B_\mu \sin(\Omega_1 t) \quad (11.151)$$

We see that the perturbations around the x_2 axis and the x_3 axis oscillate around the equilibrium values of $\lambda = \mu = 0$. We thus conclude that the rotation around the x_1 axis is stable.

Similar calculations can be done for rotations around the x_2 axis and the x_3 axis. The perturbation frequencies obtained in those cases are equal to

$$\Omega_2 = \omega_2 \sqrt{\frac{(I_1 - I_2)(I_3 - I_2)}{I_3 I_1}} \quad (11.152)$$

$$\Omega_3 = \omega_3 \sqrt{\frac{(I_2 - I_3)(I_1 - I_3)}{I_1 I_2}} \quad (11.153)$$

We see that Ω_2 is an imaginary number while Ω_3 is a real number. Thus, the rotation around the x_3 axis is stable, but the rotation around the x_2 axis is unstable.

11.8.1 Example: Problem 11.34

Consider a symmetrical rigid body rotating freely about its center of mass. A frictional torque $N_f = -b\omega$ acts to slow down the rotation. Find the component of the angular velocity along the symmetry axis as a function of time.

The Euler equation, which describes the rotation of an object about its symmetry axis, say the x axis, is

$$I_x \dot{\omega}_x - (I_y - I_z) \omega_y \omega_z = N_x \quad (11.154)$$

where $N_x = -b\omega_x$ is the component of torque along the x axis. Because the object is symmetric about the x axis, we have $I_y = I_z$, and the above equation becomes

$$I_x \frac{d\omega_x}{dt} = -b\omega_x \Rightarrow \omega_x = e^{-\frac{b}{I_x} t} \omega_{x0} \quad (11.155)$$