6 Some Methods in the Calculus of Variations

In this Chapter, we focus on an important method of solving certain problems in Classical Mechanics. In many problems we need to determine how a system evolves between an initial state and a final state. For example, consider two locations on a two-dimensional plane. Consider an object moving from one location to the other, and assume the object experiences a friction force when it moves over the surface. One can ask questions such as "What is the path between the initial and the final condition that minimizes the work done by the friction force?". Such questions can be most easily answered using the calculus of variations. The evolution that can be studied using the calculus of variations is not limited to evolutions in real space. For example, one can consider the evolution of a gas in a pV diagram and ask "What is the path between the initial and final state that maximizes the work done by the gas?"

6.1 Euler's Equation

Consider the two-dimensional plane shown in Fig. 6.1. Our initial position is specified by (x_1,y_1) and the final position is specified by (x_2,y_2). We are asked to minimize the path integral of a function f between position 1 and position 2. Suppose the path integral is minimized when we use path $y(x)$. If we change the path slightly by adding a second function $n(x)$ then we expect that the path integral to increase. Consider the following path:

$$y(\alpha, x) = y(0, x) + \alpha\eta(x) \tag{6.1}$$

The function $\eta(x)$ is an arbitrary function of x and is used to make small changes to the path. The only requirements of $\eta(x)$ are that $\eta(x)$ has a continuous first derivative and that $\eta(x)$ vanishes at the end points of the path, that is $\eta(x_1) = \eta(x_2) = 0$. Since the path integral is minimized when we follow the path, the path integral must have an extreme value when $\alpha = 0$. This requires that

$$\left.\frac{\partial J}{\partial \alpha}\right|_{\alpha=0} = \left.\frac{\partial}{\partial \alpha}\int_{x_1}^{x_2} f\left(y(\alpha,x), y'(\alpha,x); x\right) dx\right|_{\alpha=0} = 0 \tag{6.2}$$

The left-hand side of this equation can be rewritten by differentiating the argument of the integral with respect to α:

$$\left.\int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha}\right) dx\right|_{\alpha=0} = 0 \tag{6.3}$$

Using our definition of $y(\alpha, x)$ we can easily show that

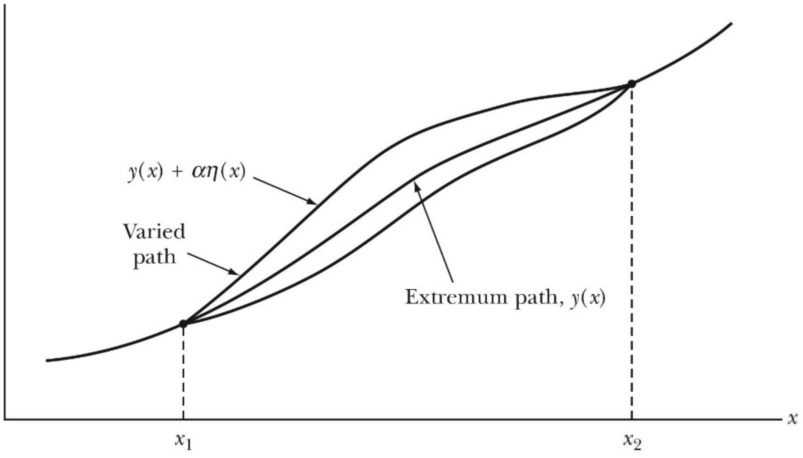

Figure 6.1: Path $y(x)$ that is used to move from position 1 to position 2.

$$\frac{\partial y}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left(y\left(0,x\right)+\alpha\eta\left(x\right)\right) = \eta\left(x\right) \tag{6.4}$$

and

$$\frac{\partial y'}{\partial \alpha} = \frac{\partial}{\partial \alpha}\frac{dy}{dx} = \frac{\partial}{\partial \alpha}\left(\frac{dy\left(0,x\right)}{dx}+\alpha\frac{d\eta\left(x\right)}{dx}\right) = \frac{d\eta\left(x\right)}{dx} \tag{6.5}$$

The requirement that the path integral has an extreme is equivalent to requiring that

$$\int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}\eta\left(x\right)+\frac{\partial f}{\partial y'}\frac{d\eta\left(x\right)}{dx}\right)dx\bigg|_{\alpha=0} = 0 \tag{6.6}$$

In order to simplify this equation we use the following relation to rewrite the second time in the integrant:

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\eta(x)\right) = \left(\frac{\partial f}{\partial y'}\frac{d\eta}{dx}\right)+\left[\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\eta(x) \tag{6.7}$$

or

$$\left(\frac{\partial f}{\partial y'}\frac{d\eta}{dx}\right) = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\eta(x)\right)-\left[\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\eta(x) \tag{6.8}$$

The second term in Eq. 6.6 can now be rewritten as

$$\begin{aligned} \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y'}\frac{d\eta(x)}{dx}\right)dx\bigg|_{\alpha=0} &= \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y'}d\eta\left(x\right)\right)\bigg|_{\alpha=0} = \\ &= \frac{\partial f}{\partial y'}\eta\left(x\right)\bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2}\left(\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta\left(x\right)\right)dx = \\ &= -\int_{x_1}^{x_2}\left(\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta\left(x\right)\right)dx \end{aligned} \tag{6.9}$$

The path integral is an extreme if

$$\int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}\eta\left(x\right)-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta\left(x\right)\right)dx\bigg|_{\alpha=0} = \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right)\eta\left(x\right)dx\bigg|_{\alpha=0} = 0 \tag{6.10}$$

Since $\eta(x)$ is an arbitrary function, this equation can only satisfied if

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \tag{6.11}$$

Equation 6.11 is known as **Euler's equation**.

6.1.1 Example: Problem 6.4

Show that the geodesic on the surface of a right-circular cylinder is a segment of a helix.

The element of distance along the surface of a cylinder is

$$dS = \sqrt{(dx)^2+(dy)^2+(dz)^2} \tag{6.12}$$

In cylindrical coordinates, (x, y, z) is related to (ρ, ϕ, z) by

$$\left.\begin{aligned} x &= \rho\cos\phi \\ y &= \rho\sin\phi \\ z &= z \end{aligned}\right| \tag{6.13}$$

where ρ is the radius of the cylinder, which is constant.
Since we consider motion on the surface of a cylinder, the radius r is constant. The expression for x, y, and z can be used to express dx, dy, and dz in cylindrical coordinates:

$$\left.\begin{aligned} dx &= -\rho\sin\phi d\phi \\ dy &= \rho\cos\phi d\phi \\ dz &= dz \end{aligned}\right] \tag{6.14}$$

Substituting Eq. Ref. (6.14) into Ref. (6.12) and integrating along the entire path, we find

$$S = \int_1^2 \sqrt{\rho^2 \ (d\phi)^2 + (dz)^2} = \int_{\phi_1}^{\phi_2} \sqrt{\rho^2 + \left(\frac{dz}{d\phi}\right)^2} \ d\phi = \int_{\phi_1}^{\phi_2} \sqrt{\rho^2 + \dot{z}^2} \ d\phi \tag{6.15}$$

If S is to be a minimum, $f \equiv \sqrt{\rho^2 + \dot{z}^2}$ must satisfy the Euler equation:

$$\frac{\partial f}{\partial z} - \frac{\partial}{\partial \phi}\frac{\partial f}{\partial \dot{z}} = 0 \tag{6.16}$$

Since $\frac{\partial f}{\partial z} = 0$, the Euler equation becomes

$$\frac{\partial}{\partial \phi}\frac{\dot{z}}{\sqrt{\rho^2 + \dot{z}^2}} = 0 \tag{6.17}$$

This condition will be satisfied if

$$\frac{\dot{z}}{\sqrt{\rho^2 + \dot{z}^2}} = \text{constant} \equiv C \tag{6.18}$$

or

$$\dot{z} = \sqrt{\frac{C^2}{1 - C^2}}\rho \tag{6.19}$$

Since ρ is constant, Ref. (6.19) implies that

$$\frac{dz}{d\phi} = \text{constant} \tag{6.20}$$

and for any point along the path, z and ϕ change at the same rate. The curve described by this condition is a *helix*.

6.1.2 Example: Problem 6.7

Consider light passing from one medium with index of refraction n_1 into another medium with index of refraction n_2 (see Fig. 6.2). Using Fermat's principle to minimize time, derive the law of refraction: $n_1\sin\theta_1 = n_2\sin\theta_2$.

The time to travel the path shown in Fig. 6.2 is

$$t = \int \frac{ds}{v} = \int \frac{\sqrt{(dx)^2 + (dy)^2}}{v} = \int \frac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{v} dx = \int \frac{\sqrt{1 + y'^2}}{v} \ dx \tag{6.21}$$

The velocity v is v_1 when $x > 0$ and v_2 when $x < 0$. The function f is given by

$$f\left(y, y'; x\right) = \frac{\sqrt{1 + y'^2}}{v} \tag{6.22}$$

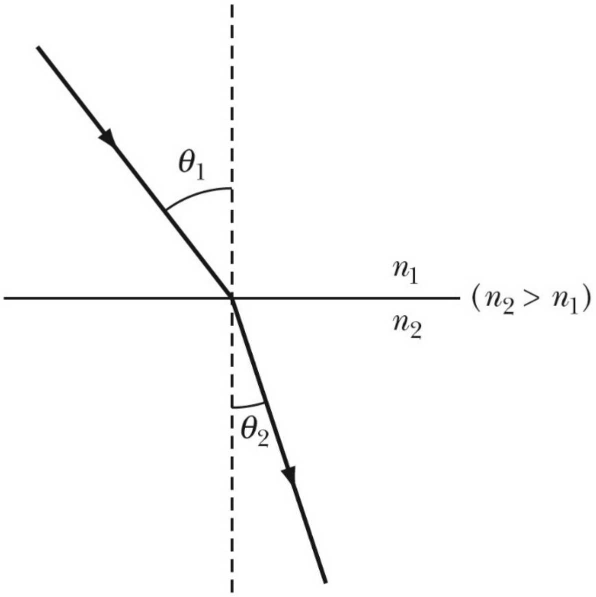

Figure 6.2: Problem 6.7. We will use the dashed line as our x axis and the interface between the two media as our y axis.

Since the velocity is not a function of y, $\partial f/\partial y = 0$. The Euler equation tells us

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = -\frac{d}{dx}\left[\frac{y'}{v\sqrt{1+y'^2}}\right] = 0 \tag{6.23}$$

Equation 6.23 requires that

$$\left[\frac{y'}{v\sqrt{1+y'^2}}\right] = \text{constant} \tag{6.24}$$

Looking at Fig. 6.2 we find that $y = x\tan\theta$ and $y' = \tan\theta$. Using $v = c/n$ we can rewrite Eq. 6.24 as

$$\frac{y'}{v\sqrt{1+y'^2}} = \frac{\tan\theta}{\left(\frac{c}{n}\right)\sqrt{1+\tan^2\theta}} = \frac{\tan\theta}{\left(\frac{c}{n}\right)\sqrt{1+\tan^2\theta}} = \frac{\tan\theta}{\left(\frac{c}{n}\right)\sqrt{1+\frac{\sin^2\theta}{\cos^2\theta}}} = \left(\frac{n}{c}\right)\sin\theta = \text{constant} \tag{6.25}$$

This proves the assertion.

6.2 Second Form of Euler's Equation

In some applications, the function f may not depend explicitly on x: $\partial f/\partial x = 0$. In order to benefit from this constraint, it would be good to try to rewrite Euler's equation with a term $\partial f/\partial x$ instead of a term $\partial f/\partial y$.

The first step in this process is to examine df/dx. The general expression for df/dx is

$$\frac{df}{dx} = \frac{d}{dx}f\left(y, y' : x\right) = \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} + \frac{\partial f}{\partial x} = y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} + \frac{\partial f}{\partial x} \tag{6.26}$$

Note that we have not assumed that $\partial f/\partial x = 0$ in order to derive this expression for df/dx. This equation can be rewritten as

$$y'' \frac{\partial f}{\partial y'} = \frac{df}{dx} - \frac{\partial f}{\partial x} - y' \frac{\partial f}{\partial y} \tag{6.27}$$

We also know that

$$\frac{d}{dx}\left(y' \frac{\partial f}{\partial y'}\right) = y'' \frac{\partial f}{\partial y'} + y' \frac{d}{dx}\frac{\partial f}{\partial y'} = \left(\frac{df}{dx} - y' \frac{\partial f}{\partial y} - \frac{\partial f}{\partial x}\right) + y' \frac{d}{dx}\frac{\partial f}{\partial y'} = \frac{df}{dx} - \frac{\partial f}{\partial x} + y'\left(\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y}\right) \tag{6.28}$$

Applying Euler's theorem to the term in the parenthesis on the right hand side, we can rewrite this equation as

$$\frac{d}{dx}\left(y' \frac{\partial f}{\partial y'}\right) = \frac{df}{dx} - \frac{\partial f}{\partial x} \tag{6.29}$$

or

$$\frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y' \frac{\partial f}{\partial y'}\right) = 0 \tag{6.30}$$

This equation is called **the second form of Euler's equation**. If f does not depend explicitly on x, $\partial f/\partial x = 0$, we conclude that

$$f - y' \frac{\partial f}{\partial y'} = \text{constant} \tag{6.31}$$

6.2.1 Example: Problem 6.4 – Part II

Show that the geodesic on the surface of a right-circular cylinder is a segment of a helix.

We have already solved this problem using the "normal" form of Euler's equation. However, looking back at the solution, we realize that the expression for f, $f \equiv \sqrt{\rho^2 + \dot{z}^2}$, does not depend explicitly on ϕ. We thus should also be able to use the second form of Euler's equation to solve this problem.

$$f - \dot{z}\frac{\partial f}{\partial \dot{z}} = \sqrt{\rho^2 + \dot{z}^2} - \dot{z}\frac{\dot{z}}{\sqrt{\rho^2 + \dot{z}^2}} = \frac{\rho^2}{\sqrt{\rho^2 + \dot{z}^2}} = \text{constant} = C \tag{6.32}$$

This equation requires that

$$\dot{z} = \frac{dz}{d\phi} = \sqrt{\frac{\rho^4}{C^2} - \rho^2} = \rho\sqrt{\frac{\rho^2}{C^2} - 1} = \text{constant} \tag{6.33}$$

Since ρ is constant, this equation implies that

$$\frac{dz}{d\phi} = \text{constant} \tag{6.34}$$

and for any point along the path, z and ϕ change at the same rate. The curve described by this condition is a *helix*.

6.3 Euler's Equation with Several Dependent Variables

Consider a situation where the function f depends on several dependent variables $y_1, y_2, y_3, \ldots$ etc., each of which depends on the independent variable x. Each dependent variable $y_i(\alpha, x)$ is related to the solution $y_i(0, x)$ in the following manner:

$$y_i(\alpha,x) = y_i(0,x) + \alpha\eta_i(x) \tag{6.35}$$

If the dependent variables y_1, y_2, y_3, etc. minimize the path integral of f, they must satisfy the following condition:

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i{}'}\right) = 0 \tag{6.36}$$

for $i = 1, 2, 3, \ldots$. The procedure to find the optimum paths is similar to the procedures we have discussed already, except that we need to solve the Euler equation for each dependent variable y_i.

6.4 Euler's Equation with Boundary Conditions

In many cases, the dependent variable y must satisfy certain boundary conditions. For example, in problem 6.4 the function y must be located on the surface of the cylinder. In this case, any point on y must satisfy the following condition:

$$r = \rho = \text{constant} \tag{6.37}$$

In general, we can specify the constraint on the path(s) by using one or more functions g and requiring that $g\{y_i; x\} = 0$.

Let us start with the case where we have two dependent variables y and z. In this case, we can write the function f as

$$f = f\{y, y', z, z'; x\} \tag{6.38}$$

In this case, we can write the differential of J with respect to α as

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left\{\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right)\eta_y(x) + \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right)\right)\eta_z(x)\right\} dx \tag{6.39}$$

For our choice of constraint, we can immediately see that the derivative of g must be zero:

$$\frac{dg}{d\alpha} = \left(\frac{\partial g}{\partial y}\right)\frac{\partial y}{\partial \alpha} + \left(\frac{\partial g}{\partial z}\right)\frac{\partial z}{\partial \alpha} = 0 \tag{6.40}$$

Using our definition of y and z we can rewrite this equation as

$$\frac{dg}{d\alpha} = \left(\frac{\partial g}{\partial y}\right)\eta_y + \left(\frac{\partial g}{\partial z}\right)\eta_z = 0 \tag{6.41}$$

This equation shows us a general relation between the functions η_z and η_z:

$$\frac{\eta_z}{\eta_y} = -\frac{\left(\frac{\partial g}{\partial y}\right)}{\left(\frac{\partial g}{\partial z}\right)} \tag{6.42}$$

Using this relation, we can rewrite our expression for the differential of J:

$$\begin{aligned}\frac{\partial J}{\partial \alpha} &= \int_{x_1}^{x_2}\left\{\left(\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right)\eta_y(x)+\left(\frac{\partial f}{\partial z}-\frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right)\right)\eta_z(x)\right\}dx = \\ &= \int_{x_1}^{x_2}\left\{\left(\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right)+\left(\frac{\partial f}{\partial z}-\frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right)\right)\frac{\eta_z(x)}{\eta_y(x)}\right\}\left\{\eta_y(x)\right\}dx \\ &= \int_{x_1}^{x_2}\left\{\left(\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right)-\left(\frac{\partial f}{\partial z}-\frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right)\right)\frac{\left(\frac{\partial g}{\partial y}\right)}{\left(\frac{\partial g}{\partial z}\right)}\right\}\left\{\eta_y(x)\right\}dx\end{aligned} \tag{6.43}$$

Since the function η_y is an arbitrary function, the equation can only evaluate to 0 if the term in the brackets is equal to 0:

$$\left(\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right)-\left(\frac{\partial f}{\partial z}-\frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right)\right)\frac{\left(\frac{\partial g}{\partial y}\right)}{\left(\frac{\partial g}{\partial z}\right)}=0 \tag{6.44}$$

This equation can be rewritten as

$$\left(\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right)\left(\frac{\partial g}{\partial y}\right)^{-1}=\left(\frac{\partial f}{\partial z}-\frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right)\right)\left(\frac{\partial g}{\partial z}\right)^{-1} \tag{6.45}$$

This equation can only be correct if both sides are equal to a function that depends only on x:

$$\left(\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right)\left(\frac{\partial g}{\partial y}\right)^{-1}=\left(\frac{\partial f}{\partial z}-\frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right)\right)\left(\frac{\partial g}{\partial z}\right)^{-1}=-\lambda(x) \tag{6.46}$$

or

$$\left(\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right)+\lambda(x)\left(\frac{\partial g}{\partial y}\right)=0 \tag{6.47}$$

$$\left(\frac{\partial f}{\partial z}-\frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right)\right)+\lambda(x)\left(\frac{\partial g}{\partial z}\right)=0 \tag{6.48}$$

Note that in this case, where we have one auxiliary condition, $g\{y, z; x\} = 0$, we end up with one Lagrange undetermined multiplier $\lambda(x)$. Since we have three equations and three unknown (y, z, and λ), we can determine the unknown.

In certain problems, the constraint can only be written in integral form. For example, the constraint for problems dealing with ropes will be that the total length of the path is equal to the length of the rope L:

$$K[y]=\int g\{y,y';x\}\,dx=L \tag{6.49}$$

The curve y then must satisfy the following differential equation:

$$\left(\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right)+\lambda(x)\left(\frac{\partial g}{\partial y}-\frac{d}{dx}\left(\frac{\partial g}{\partial y'}\right)\right)=0 \tag{6.50}$$

6.4.1 Problem 6.12

Repeat example 6.4, finding the shortest path between any two points on the surface of a sphere, but use the method of the Euler equation with an auxiliary condition imposed.

The path length is given by

$$s=\int ds=\int\sqrt{1+y'^2+z'^2}\;dx \tag{6.51}$$

The function f is thus equal to

$$f(y,y',z,z';x) = \sqrt{1+y'^2+z'^2} \tag{6.52}$$

and our equation of constraint is

$$g(x,y,z) = x^2+y^2+z^2-\rho^2 = 0 \tag{6.53}$$

The Euler equations with undetermined multipliers, Eq. 6.47 and Eq. 6.48, tell us that

$$\frac{d}{dx}\left[\frac{y'}{\sqrt{1+y'^2+z'^2}}\right] = \lambda\frac{dg}{dy} = 2\lambda y \tag{6.54}$$

and

$$\frac{d}{dx}\left[\frac{z'}{\sqrt{1+y'^2+z'^2}}\right] = \lambda\frac{dg}{dz} = 2\lambda z \tag{6.55}$$

Eliminating the factor λ, we obtain

$$\frac{1}{y}\frac{d}{dx}\left[\frac{y'}{\sqrt{1+y'^2+z'^2}}\right] - \frac{1}{z}\frac{d}{dx}\left[\frac{z'}{\sqrt{1+y'^2+z'^2}}\right] = 0 \tag{6.56}$$

This simplifies first to

$$z\left[y''\left(1+y'^2+z'^2\right)-y'(y'y''+z'z'')\right]-y\left[z''\left(1+y'^2+z'^2\right)-z'(y'y''+z'z'')\right] = 0 \tag{6.57}$$

and then to

$$zy''+(yy'+zz')z'y''-yz''-(yy'+zz')y'z'' = 0 \tag{6.58}$$

Using the derivative of Ref. (6.53) we can rewrite Eq. 6.58 as

$$(z-xz')y'' = (y-xy')z'' \tag{6.59}$$

This looks to be in the simplest form we can make it, but is it a plane? Take the equation of a plane passing through the origin:

$$Ax+By = z \tag{6.60}$$

and make it a differential equation by taking derivatives (giving $A+By'=z'$ and $by''=z''$ and eliminating the constants:

$$B = \frac{z''}{y''} \tag{6.61}$$

$$A = z'-By' = z'-\frac{z''}{y''} \tag{6.62}$$

This substitution yields Ref. (6.59) exactly. This confirms that the path must be the intersection of the sphere with a plane passing through the origin, as required.

6.4.2 Example: Problem 6.4 – Part III

Show that the geodesic on the surface of a right circular cylinder is a segment of a helix.

We have already solved this problem using the "normal" form of Euler's equation. However, this problem is a good example of how to approach problems with constraints. **Note: doing it in this way is NOT easier than the approaches we have used previously.**

Let us consider two points on the surface of the cylinder:

$$x_i = (\rho\cos\phi_i, \rho\sin\phi_i, z_i) \tag{6.63}$$

and

$$x_f = \left(\rho\cos\phi_f, \rho\sin\phi_f, z_f\right) \tag{6.64}$$

Consider an arbitrary path connecting the initial and final position. The length of a tiny segment of this path is

$$dl = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = dz\sqrt{1 + \left(\frac{dx}{dz}\right)^2 + \left(\frac{dy}{dz}\right)^2 + 1} \tag{6.65}$$

The integral we want to minimize is

$$J = \int dl = \int \left(\sqrt{\left(\frac{dx}{dz}\right)^2 + \left(\frac{dy}{dz}\right)^2 + 1} \right) dz = \int \left(\sqrt{(x')^2 + (y')^2 + 1} \right) dz \tag{6.66}$$

We immediately see that z is our independent variable and that x, and y are our dependent variables. The function f is thus given by

$$f(x, x', y, y'; z) = \sqrt{(x')^2 + (y')^2 + 1} \tag{6.67}$$

The solution y is constrained to be on the surface of the cylinder, and the following equation of constraint needs to be applied:

$$g(x, y) = x^2 + y^2 - \rho^2 = 0 \tag{6.68}$$

For this equation of constraint we know that

$$\frac{\partial g}{\partial x} = 2x \tag{6.69}$$

and

$$\frac{\partial g}{\partial y} = 2y \tag{6.70}$$

Note: if we had picked our function of constraint to be

$$g(x, y) = \sqrt{x^2 + y^2} - \rho = 0 \tag{6.71}$$

we would get more complicated partial derivatives of g.

To solve the current problem, we thus need to solve the following Euler equations:

$$\left(\frac{\partial f}{\partial x} - \frac{d}{dz}\left(\frac{\partial f}{\partial x'}\right)\right) + \lambda(z)\left(\frac{\partial g}{\partial x}\right) = -\frac{d}{dz}\left(\frac{x'}{\sqrt{(x')^2 + (y')^2 + 1}}\right) + 2\lambda(z)x = 0 \tag{6.72}$$

$$\left(\frac{\partial f}{\partial y}-\frac{d}{dz}\left(\frac{\partial f}{\partial y'}\right)\right)+\lambda(z)\left(\frac{\partial g}{\partial y}\right)=-\frac{d}{dz}\left(\frac{y'}{\sqrt{(x')^2+(y')^2+1}}\right)+2\lambda(z)y=0 \tag{6.73}$$

These two equations can be rewritten as

$$y\frac{d}{dz}\left(\frac{x'}{\sqrt{(x')^2+(y')^2+1}}\right)=2\lambda(z)xy \tag{6.74}$$

$$x\frac{d}{dz}\left(\frac{y'}{\sqrt{(x')^2+(y')^2+1}}\right)=2\lambda(z)xy \tag{6.75}$$

Eliminating λ we find that

$$y\frac{d}{dz}\left(\frac{x'}{\sqrt{(x')^2+(y')^2+1}}\right)=x\frac{d}{dz}\left(\frac{y'}{\sqrt{(x')^2+(y')^2+1}}\right) \tag{6.76}$$

This equation can be rewritten as

$$y\left(\frac{x''}{\sqrt{(x')^2+(y')^2+1}}-\frac{x'\{x'x''+y'y''\}}{\left((x')^2+(y')^2+1\right)^{3/2}}\right)=x\left(\frac{y''}{\sqrt{(x')^2+(y')^2+1}}-\frac{y'\{x'x''+y'y''\}}{\left((x')^2+(y')^2+1\right)^{3/2}}\right) \tag{6.77}$$

After simplifying this equation we obtain

$$y\left(x''\left(1+(y')^2\right)-x'y'y''\right)=x\left(y''\left(1+(x')^2\right)-x'y'x''\right) \tag{6.78}$$

Is this equation describing a helix? Yes, it is! How do you see that? Let us look at the definition of a helix:

$$\begin{aligned} x &= \rho\cos\phi \\ y &= \rho\sin\phi \\ z &= \beta\phi \end{aligned} \tag{6.79}$$

We find that

$$\begin{aligned} x' &= \tfrac{dx}{dz}=\tfrac{dx}{d\phi}\tfrac{d\phi}{dz}=\tfrac{1}{\beta}(-\rho\sin\phi)=-\tfrac{1}{\beta}y \\ y' &= \tfrac{dy}{dz}=\tfrac{dy}{d\phi}\tfrac{d\phi}{dz}=\tfrac{1}{\beta}(\rho\cos\phi)=\tfrac{1}{\beta}x \end{aligned} \tag{6.80}$$

and

$$\begin{aligned} x'' &= -\tfrac{1}{\beta}y'=-\tfrac{1}{\beta^2}x \\ y'' &= \tfrac{1}{\beta}x'=-\tfrac{1}{\beta^2}y \end{aligned} \tag{6.81}$$

Taking these relations and substituting them in the solution shown in Eq. 6.78 we find:

$$y\left(x''\left(1+(y')^2\right)-x'y'y''\right)=-\frac{1}{\beta^2}xy-\frac{1}{\beta^4}x^3y-\frac{1}{\beta^4}xy^3=x\left(y''\left(1+(x')^2\right)-x'y'x''\right) \tag{6.82}$$

6.5 The δ notation

It is common to use the δ notation in the calculus of variations. In order to use the δ notation, we use the following definitions:

$$\delta y = \frac{\partial y}{\partial \alpha} d\alpha \tag{6.83}$$

$$\delta J = \frac{\partial J}{\partial \alpha} d\alpha \tag{6.84}$$

In terms of these variables, we find

$$\begin{aligned} \delta J &= \tfrac{\partial J}{\partial \alpha} d\alpha = \left\{ \int_{x_1}^{x_2} \left(\tfrac{\partial f}{\partial y} - \tfrac{d}{dx}\left(\tfrac{\partial f}{\partial y'} \right) \right) \tfrac{\partial y}{\partial \alpha} dx \right\} d\alpha = \\ &= \int_{x_1}^{x_2} \left(\tfrac{\partial f}{\partial y} - \tfrac{d}{dx}\left(\tfrac{\partial f}{\partial y'} \right) \right) \tfrac{\partial y}{\partial \alpha} d\alpha dx = \\ &= \int_{x_1}^{x_2} \left(\tfrac{\partial f}{\partial y} - \tfrac{d}{dx}\left(\tfrac{\partial f}{\partial y'} \right) \right) \delta y dx \end{aligned} \tag{6.85}$$

Since δy is an arbitrary function, the requirement that $\delta J = 0$ requires that the term in parentheses be 0. This, of course, is the Euler equation we have encountered before!

It is important to note that there is a significant difference between δy and dy. Based on the definition of δy we see that δy tells us how y varies when we change α while keeping all other variables fixed (including, for example, time t).