Do not turn the pages of the exam until you are instructed to do so.

Exam rules: You may use **only** a writing instrument while taking this test. You may **not** consult any calculators, computers, books, nor each other.

Problems 1, 2, and 3 must be answered in exam booklet 1. Problems 4, 5, and 6 must be answered in exam booklet 2. Problem 7 and 8 must be answered in exam booklet 3. Your answers to Problems 1, 2, 3, 5, 6, 7, and 8 must be well motivated and expressed in terms of the variables used in the problem. You will receive partial credit where appropriate, but only if we can read your solution. Answers that are not motivated will not receive any credit, even if correct. No justification is required for your answers to Problem 4, although you are of course free to provide it.

At the end of the exam, you must hand in your exam with the honor pledge completed, the two blue exam booklets, and the equation sheet. All items must be clearly labeled with your name, your student ID number. **If any of these items are missing, we will not grade your exam, and you will receive a score of 0 points.**

You are required to complete the following *Honor Pledge for Exams*. Copy and sign the pledge before starting your exam.

"I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own."

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Name: ____________________________________

Signature: ____________________________________

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$$
\begin{array}{ccc}
\cos(30°) = \frac{1}{2}\sqrt{3} & \sin(30°) = \frac{1}{2} & \tan(30°) = \frac{1}{3}\sqrt{3} \\
\cos(45°) = \frac{1}{2}\sqrt{2} & \sin(45°) = \frac{1}{2}\sqrt{2} & \tan(45°) = 1 \\
\cos(60°) = \frac{1}{2} & \sin(60°) = \frac{1}{2}\sqrt{3} & \tan(60°) = \sqrt{3}
\end{array} \tag{1}
$$

$$
\begin{array}{ccc}
 & \text{Circle} & \text{Sphere} \\
\text{circumference} & 2\pi r & \\
\text{(surface) area} & \pi r^2 & 4\pi r^2 \\
\text{volume} & & \frac{4}{3}\pi r^3
\end{array} \tag{2}
$$

$$\int x \sin x dx = -x \cos x + \sin x \tag{3}$$

$$\int x^2 \sin x dx = -x^2 \cos x + 2x \sin x + 2 \cos x \tag{4}$$

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Good Luck !

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Problem 1 (25 points) **Answer in booklet 1**

Consider a pendulum, shown in Fig. 1, composed of a rigid rod of length b with a mass $m_1 = m$ at its end. Another mass $m_2 = m$ is placed halfway down the rod. Consider the motion of the pendulum when the oscillations are small, and assume that the pendulum swings in the plane of the Figure. Assume that the mass of the rod is 0 kg.

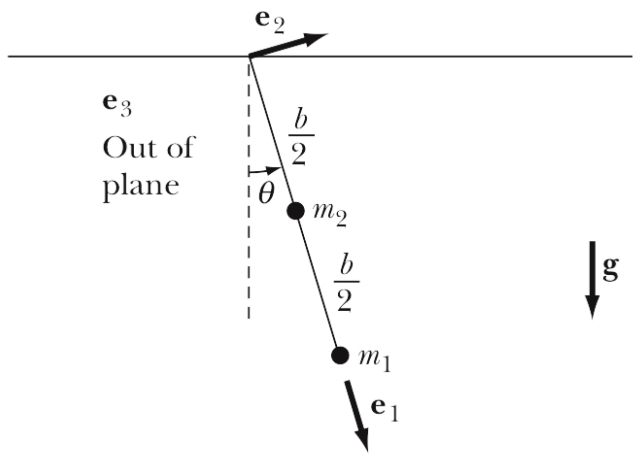

Figure 1: Problem 1

a) **(5 points)** What is the inertia tensor?

b) **(5 points)** Express the angular momentum of the pendulum in terms of its angular velocity $\dot{\theta}$.

c) **(5 points)** Use the external torque acting on the pendulum to determine the equation of motion for the angle θ.

d) **(5 points)** What is the Lagrangian of the system?

e) **(5 points)** Use the Lagrange equation of motion to determine the angular frequency of the system.

Your answers must be well motivated and expressed in terms of the variables provided.

Problem 2 (25 points) **Answer in booklet 1**

Consider an elastic collision of two particles with mass m_1 and m_2. Mass m_2 is initially at rest in the laboratory system. Two possible collisions with different impact parameters are shown in Fig. 2. Assume that particle 1 is moving with a velocity v_1 before the collision and has the same mass as particle 2 ($m_1 = m_2 = m$).

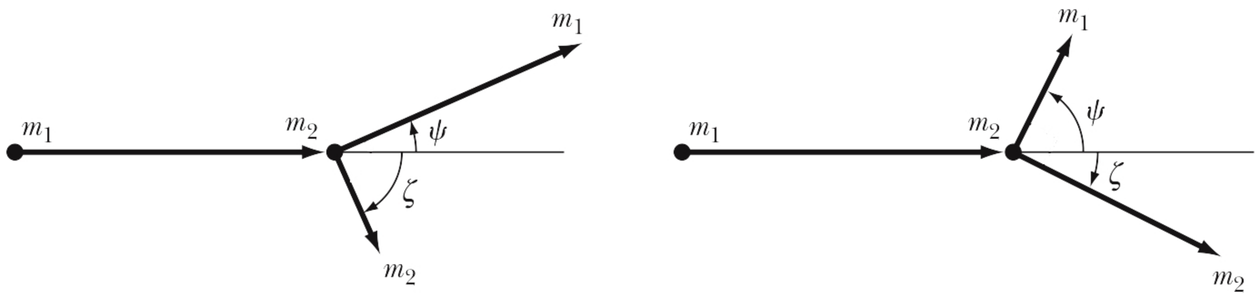

Figure 2: Two possible outcomes of the elastic scattering of two particles of equal mass as seen in the laboratory frame. Particle 2 is initially at rest in the laboratory frame.

a) **(5 points)** What are the velocities of particles 1 and 2 before the collision in the center-of-mass frame?

b) **(5 points)** The center-of-mass scattering angle of particle 1 after the collision, measured with respect to the direction of particle 1 before the collision, is θ. What are the velocities of particles 1 and 2 after the collision in the center-of-mass frame?

c) **(10 points)** What are the laboratory scattering angles ψ and ζ of particles 1 and 2, respectively, after the collision?

d) **(5 points)** Show that after the collision, the two masses travel at right angles with respect to each other in the laboratory frame.

Your answer needs to be well motivated and expressed in terms of the variables provided.

Problem 3 (25 points) **Answer in booklet 1**

Consider a string of length L, kept fixed at both ends. The string has a density ρ and a tension τ. The condition of the string at time $t = 0$ s is specified by the following boundary conditions:

$$q(x,0) = 4\frac{x(L-x)}{L^2} \tag{5}$$

and

$$\dot{q}(x,0) = 0 \tag{6}$$

a) **(20 points)** Find the characteristic frequencies.

b) **(5 points)** Calculate the amplitude of the n^{th} node.

Your answers must be well motivated and expressed in terms of the variables provided.

Problem 4 (25 points) **Answer in booklet 2**

a) **(5 points)** Consider light passing from one medium with index of refraction n_1 into another medium with index of refraction n_2, as shown in Fig. 3.

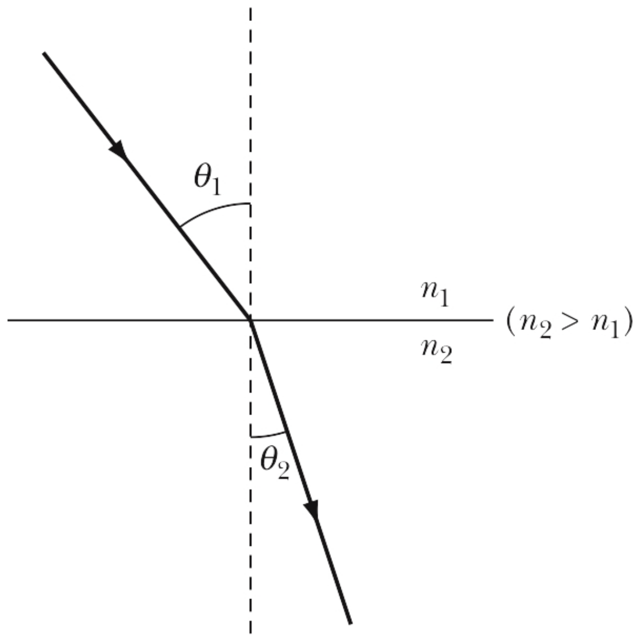

Figure 3: Problem 4a.

The x axis is defined by the interface between the two media and the y axis is perpendicular to this surface (represented by the dashed line in Fig. 3).

What would you use as your independent variable when you derive the law of refraction using one of the Euler equations?

b) **(5 points)** Consider a particle dropped from a height h above the surface of the Earth at a latitude λ. What is the direction of the deflection of this particle caused by the Coriolis force? Use the coordinate system shown in Fig. 4 to answer this question.

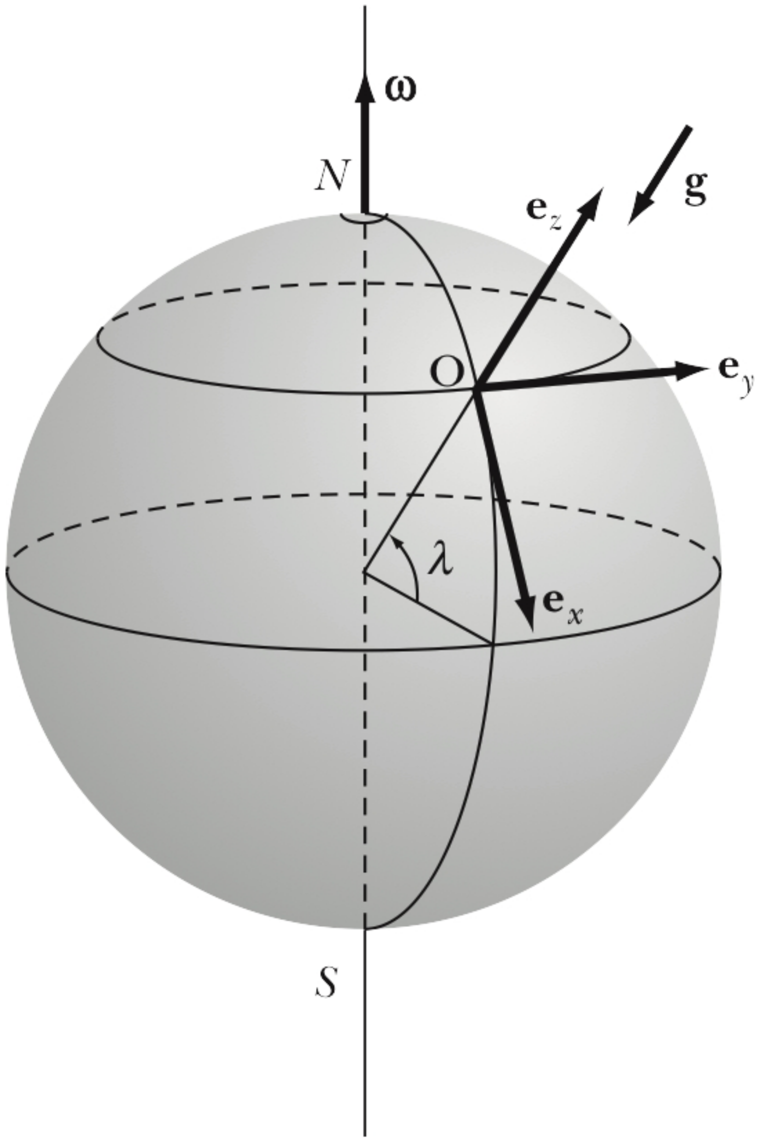

Figure 4: The coordinate system on the Earth's surface used to describe the motion of a falling particle. The vector $\hat{e}_x$ is in the southerly direction. The vector $\hat{e}_y$ is in the easterly direction.

c) **(5 points)** Which of the curves shown in Fig. 5 shows the time dependence of the amplitude of an over-damped oscillator?

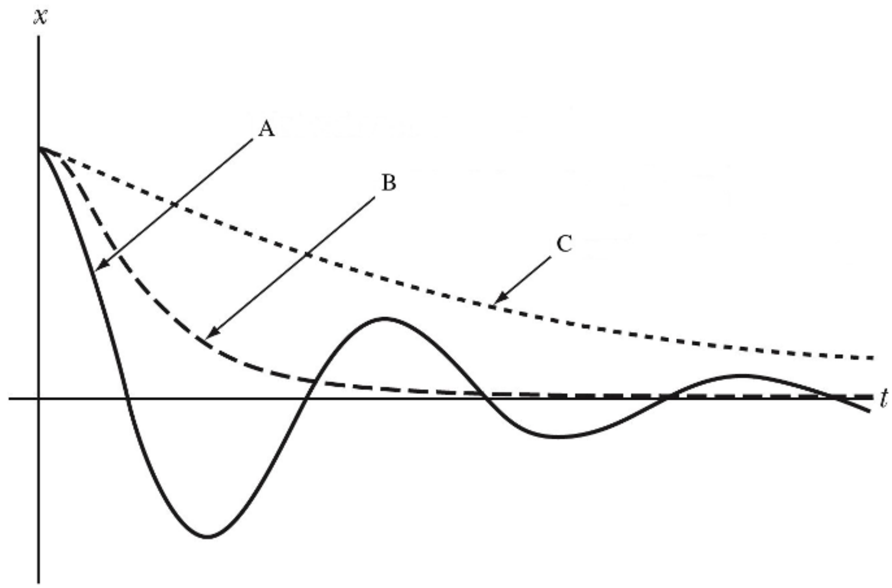

Figure 5: Problem 4c.

d) **(5 points)** Consider a disk rolling without slipping on an inclined plane, as shown in Fig. 6. Consider the coordinates y and θ.

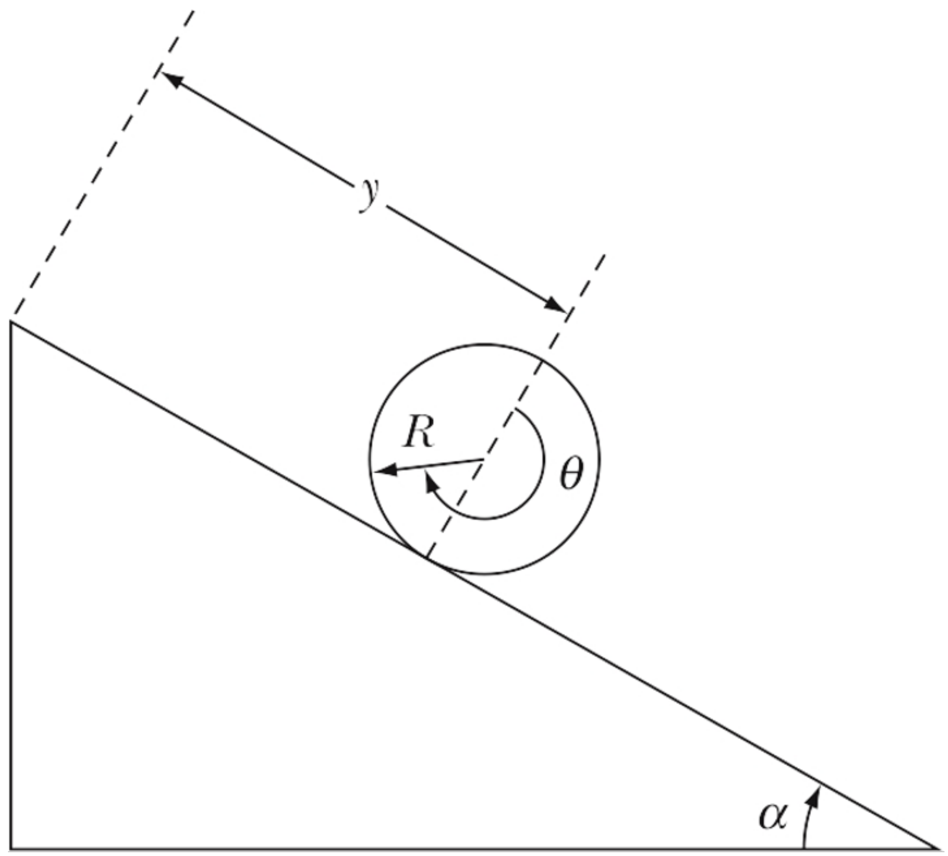

Figure 6: Problem 4d.

What is the equation of constraint you would use for this system?

e) **(5 points)** At what airport was the photograph shown in Fig. 7 made?

Figure 7: Problem 4e.

Problem 5 (25 points) **Answer in booklet 2**

Consider a particle traveling in a constant force field (e.g. the gravitational field), directed along the positive x axis. The particle starts from rest at $(0,0)$ and moves to a position (x_2, y_2), as shown in Fig. 8.

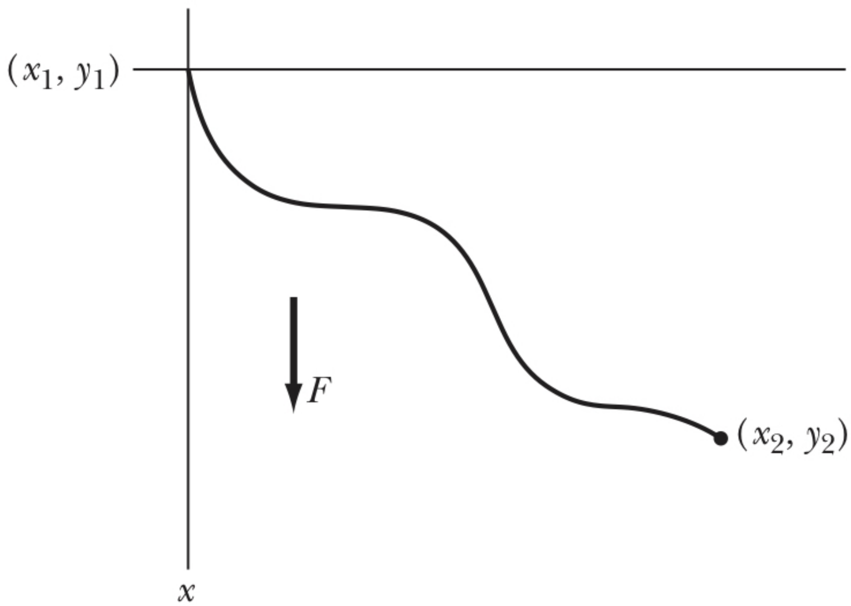

Figure 8: Problem 5.

a) **(5 points)** How does the velocity of the particle depend on x?

b) **(5 points)** Express the time it takes for the particle to move from its initial position to its final position, following a path $y(x)$, in terms of an integral over x between $x = 0$ and $x = x_2$.

c) **(15 points)** Find the path that allows the particle to complete this movement in the shortest time.

Your answers must be well motivated and expressed in terms of the variables provided.

Problem 6 (25 points) **Answer in booklet 2**

Suppose that you drilled a hole through the center of the Earth to deliver mail from one side of the Earth to the other side, as shown in Fig. 9. The Earth has a radius R_E and a mass M_E.

Figure 9: Problem 6. Figure taken from https://www.askamathematician.com/2012/08/q-if-you-could-drill-a-tunnel-through-the-whole-planet-and-then-jumped-down-this-tunnel-how-would-you-fall/

a) **(10 points)** Suppose you drop a package of mass m into the hole (at the location of the small circle in Fig. 9). What is the gravitational force acting on this package as function of the distance r from the center of the Earth? **Note: you must specify both direction and magnitude.**

b) **(15 points)** How long does it take for the package to be delivered to the other side of the Earth?

Your answers must be well motivated and expressed in terms of the variables provided.

Problem 7 (25 points) **Answer in booklet 3**

A smooth rope of length L and density ρ is placed above a hole in a table, as shown in Fig. 10. One end of the rope falls through the hole at time $t = 0$, pulling steadily on the remainder of the rope. Ignore all friction.

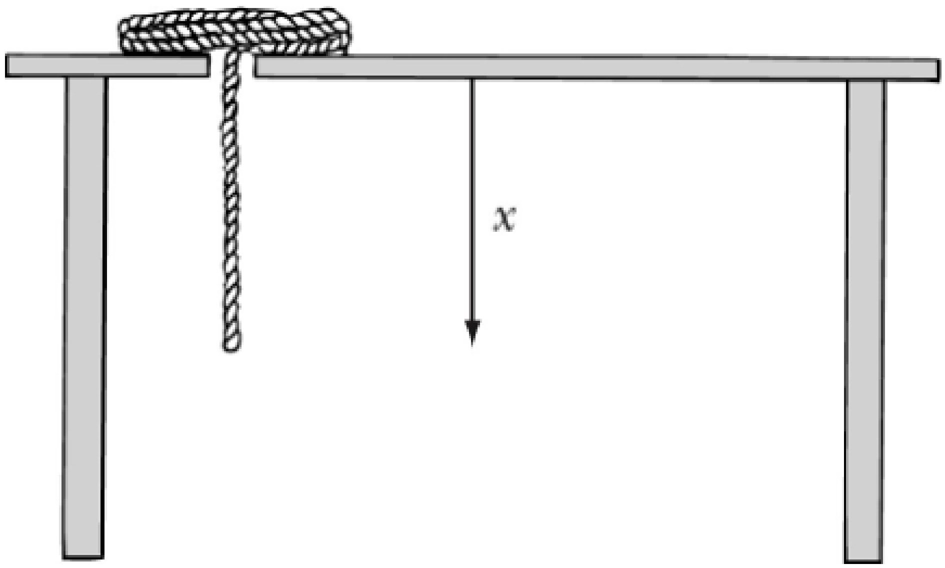

Figure 10: Problem 7.

a) **(5 points)** At time t, a length x of the rope has passed through the hole. Assume that the velocity of the rope at this time is v. Use the relation $\vec{F} = d\vec{p}/dt$ to obtain a differential equation relating x and v (and/or their derivatives).

b) **(5 points)** Solve the differential equation found in part a) by using $v = ax^n$ as a trial function.

c) **(5 points)** Determine the acceleration of the rope as function of x.

d) **(5 points)** What is the Lagrangian of this system?

e) **(5 points)** What is the Lagrange equation of motion for x? Compare this answer with your answer to part a).

Your answers must be well motivated and expressed in terms of the variables provided.

Problem 8 (25 points) **Answer in booklet 3**

A bucket of water is set spinning around its vertical symmetry axis with a constant angular velocity ω, as shown in Fig. 11.

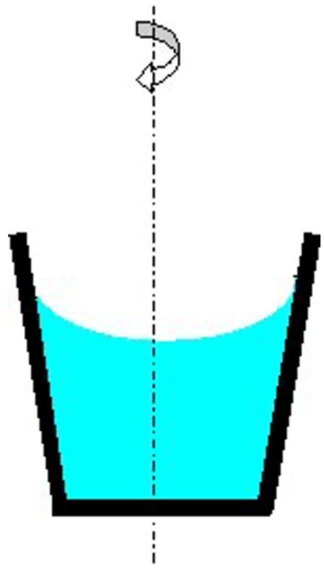

Figure 11: Problem 8.

a) **(2.5 points)** Consider a small volume of water of mass m, located on the surface of the water, a distance r from the symmetry axis. What is the effective force on this mass of water?

b) **(2.5 points)** After a while, the shape of the water surface in the bucket no longer changes. What is the effective force on the small volume of water at that time?

c) **(10 points)** What is the pressure gradient acting on this volume of water? **You can express your answer in vector notation.**

d) **(10 points)** When the system has reached an equilibrium state, what is the function $z(r)$ that describes the shape of the surface (z is the vertical displacement and r is the distance from the rotation axis)?

Your answers must be well motivated and expressed in terms of the variables provided.

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Figure 12: It is not the KLM, but nicer to fly! Your instructor is shown after his flight from Rochester to Piseco.