

Physics 121, April 8, 2008.

Harmonic Motion.



Physics 121.

April 8, 2008.

- Course Information
- Topics to be discussed today:
 - Simple Harmonic Motion (Review).
 - Simple Harmonic Motion: Example Systems.
 - Damped Harmonic Motion
 - Driven Harmonic Motion

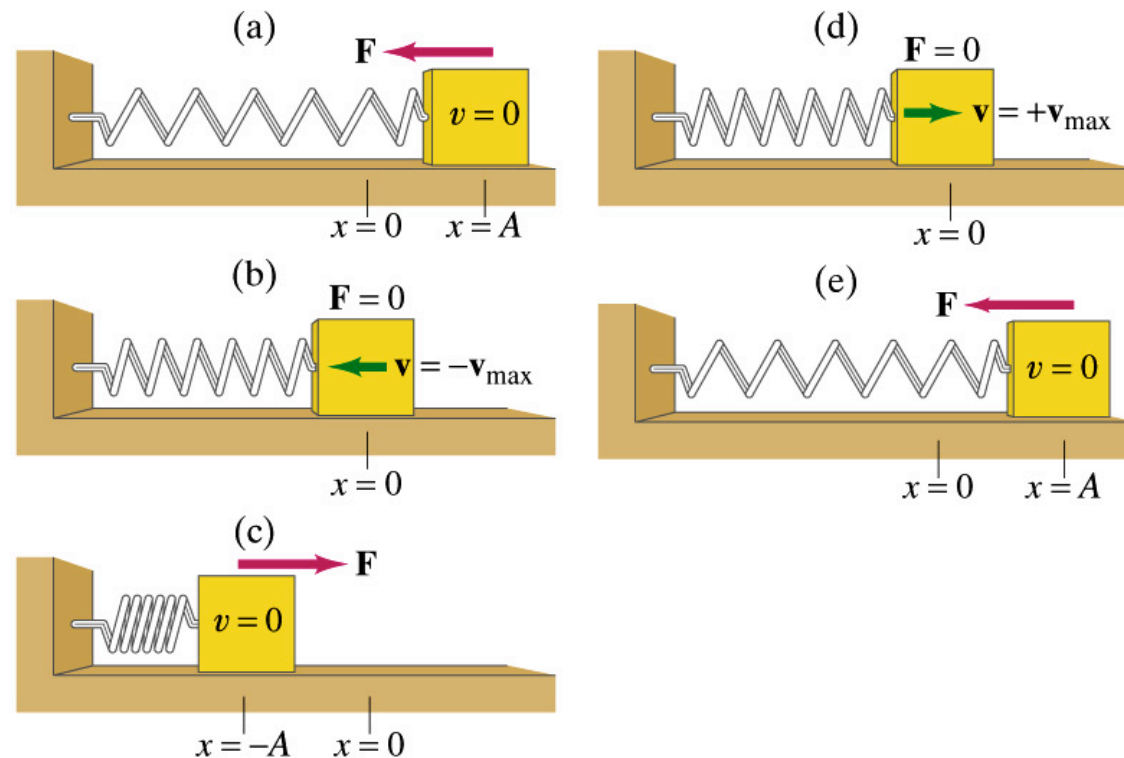
Physics 121.

April 8, 2008.

- Homework set # 8 is due on Saturday morning, April 12, at 8.30 am.
- Homework set # 9 will be available on Saturday morning at 8.30 am, and will be due on Saturday morning, April 19, at 8.30 am.
- Requests for regarding part of Exam # 1 and # 2 need to be given to me by April 17. You need to write down what I should look at and give me your written request and your blue exam booklet(s).

Harmonic motion (a quick review).

Motion that repeats itself at regular intervals.



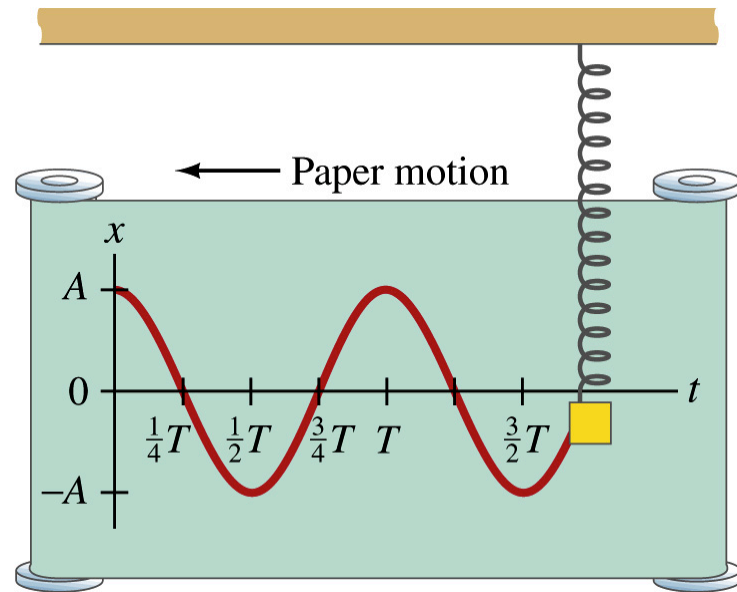
Simple Harmonic Motion (a quick review).

Phase Constant

Amplitude

$$x(t) = x_m \cos(\omega t + \phi)$$

Angular Frequency



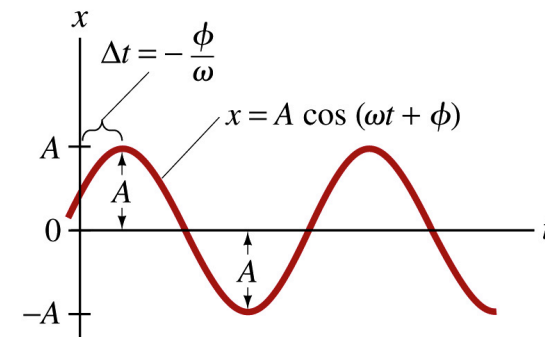
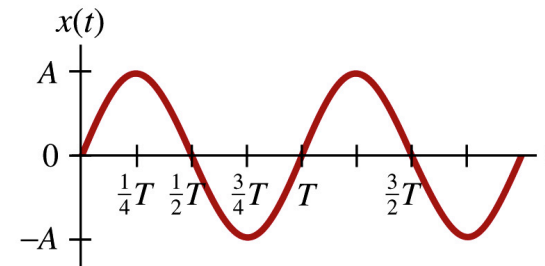
Simple Harmonic Motion (a quick review).

- Other variables frequently used to describe simple harmonic motion:

- The period T : the time required to complete one oscillation. The period T is equal to $2\pi/\omega$.
- The frequency of the oscillation is the number of oscillations carried out per second:

$$\nu = 1/T$$

The unit of frequency is the Hertz (Hz). Per definition, $1 \text{ Hz} = 1 \text{ s}^{-1}$.



Simple Harmonic Motion (a quick review).

What forces are required?

- Using Newton's second law we can determine the force responsible for the harmonic motion:

$$F = ma = -m\omega^2 x$$

- The total mechanical energy of a system carrying out simple harmonic motion is constant.
- A good example of a force that produces simple harmonic motion is the spring force: $F = -kx$. The angular frequency depends on both the spring constant k and the mass m :

$$\omega = \sqrt{k/m}$$

Simple Harmonic Motion (SHM).

The torsion pendulum.

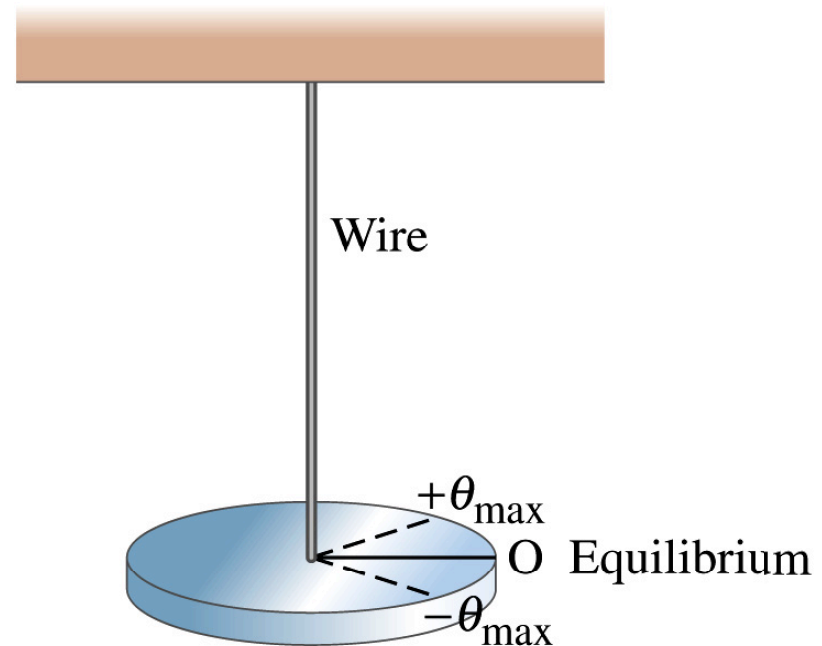
- What is the angular frequency of the SHM of a torsion pendulum:
 - When the base is rotated, it twists the wire and a torque is generated which is proportional to the angular twist:

$$\tau = -K\theta$$

The torque generates an angular acceleration α :

$$\alpha = d^2\theta/dt^2 = \tau/I = -(K/I)\theta$$

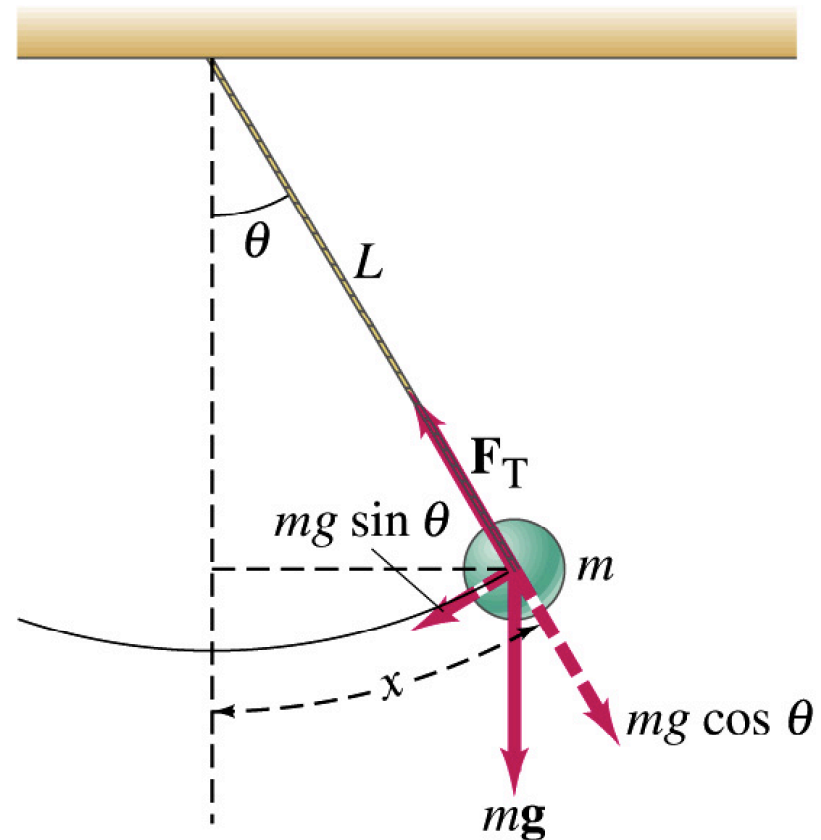
The resulting motion is harmonic motion with an angular frequency $\omega = \sqrt{K/I}$.



Simple Harmonic Motion (SHM).

The simple pendulum.

- Calculate the angular frequency of the SHM of a simple pendulum.
 - A simple pendulum is a pendulum for which all the mass is located at a single point at the end of a massless string.
 - There are two forces acting on the mass: the tension T and the gravitational force mg .
 - The tension T cancels the radial component of the gravitational force.



Simple Harmonic Motion (SHM).

The simple pendulum.

- The net force acting on the mass is directed perpendicular to the string and is equal to

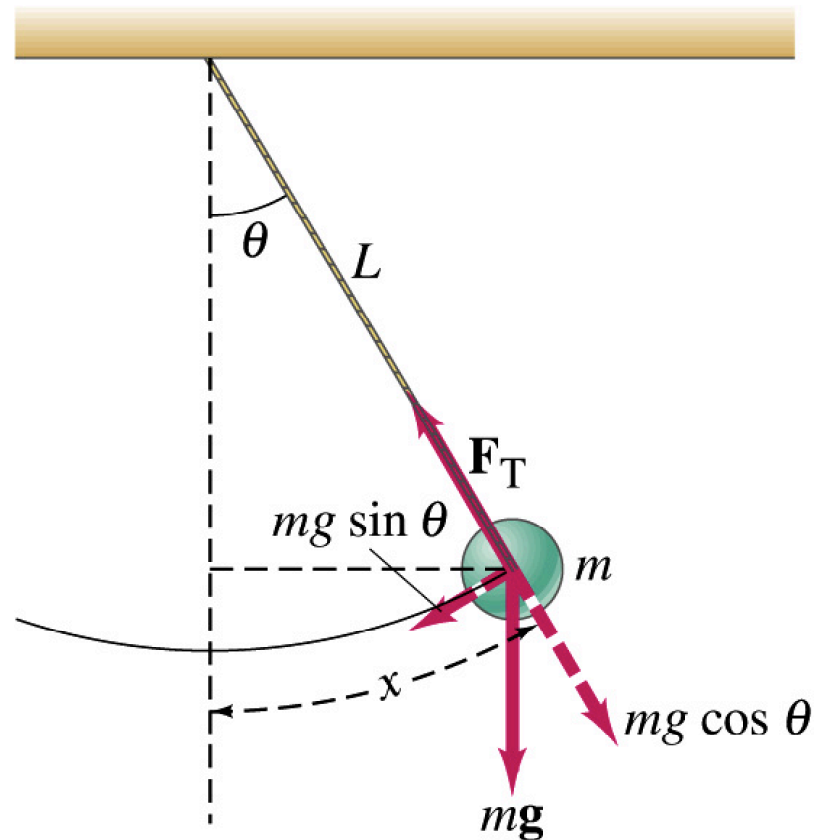
$$F = -mg \sin \theta$$

The minus sign indicates that the force is directed opposite to the angular displacement.

- When the angle θ is small, we can approximate $\sin \theta$ by θ :

$$F = -mg\theta = -mg x/L$$

- Note: the force is again proportional to the displacement.



Simple Harmonic Motion (SHM).

The Simple Pendulum.

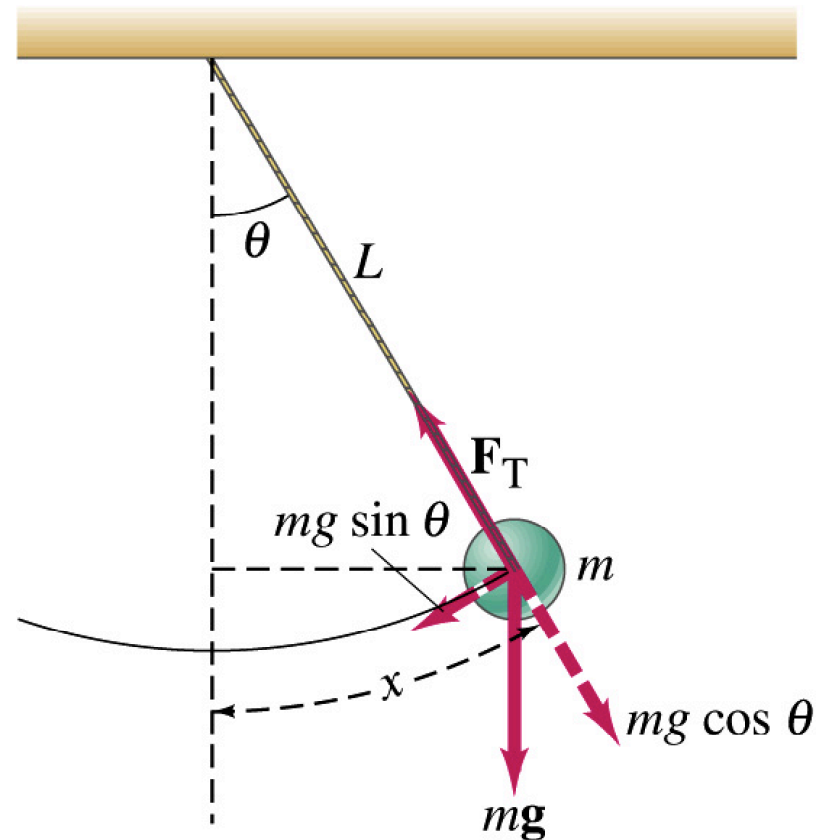
- The equation of motion for the pendulum is thus

$$F = m \, d^2x/dt^2 = -(mg/L) x$$

or

$$d^2x/dt^2 = - (g/L) x$$

- The equation of motion is the same as the equation of motion for a SHM, and the pendulum will thus carry out SHM with an angular frequency $\omega = \sqrt{g/L}$.
- The period of the pendulum is thus $2\pi/\omega = 2\pi \sqrt{L/g}$. Note: the period is independent of the mass of the pendulum.



Simple Harmonic Motion (SHM).

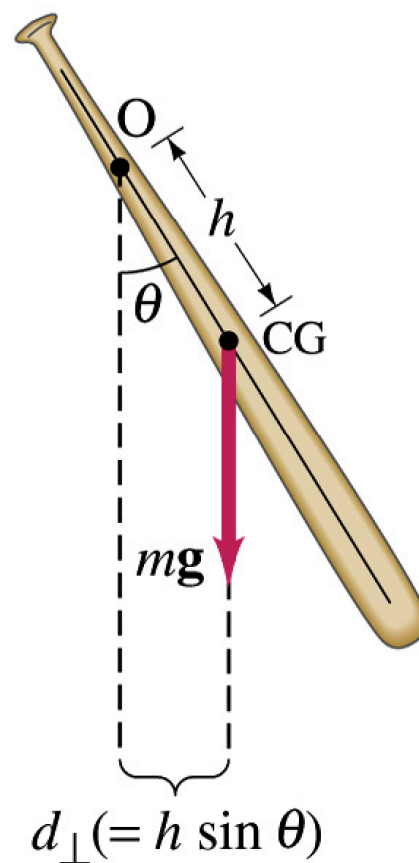
The physical pendulum.

- In a realistic pendulum, not all mass is located at a single point.
- The motion carried out by this realistic pendulum around its rotation point O can be determined by determining the total torque with respect to this point:

$$\tau = -mgh \sin \theta$$

- If the angle θ is small, we can approximate the torque by

$$\tau = -mgh\theta$$



Simple Harmonic Motion (SHM).

The physical pendulum.

- The angular acceleration α is related to the torque:

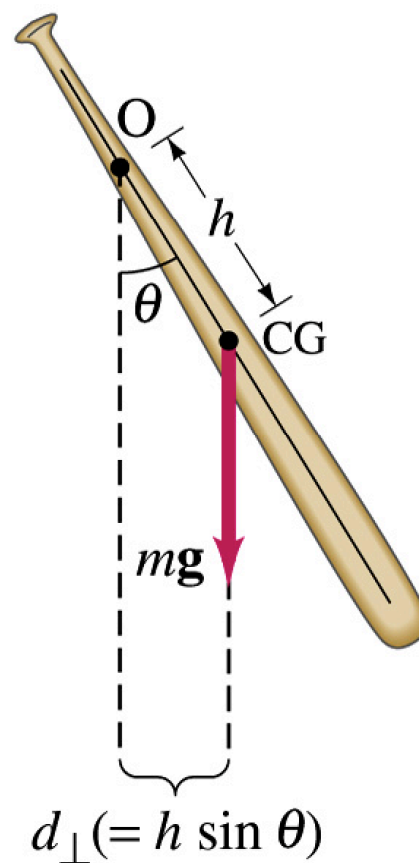
$$\tau = I\alpha$$

- The equation of motion for the angular acceleration α is given by

$$\alpha = \frac{d^2\theta}{dt^2} = \frac{\tau}{I} = -\frac{mgh}{I}\theta$$

- This again is an equation for SHM with an angular frequency ω where

$$\omega^2 = \frac{mgh}{I}$$



Simple Harmonic Motion (SHM).

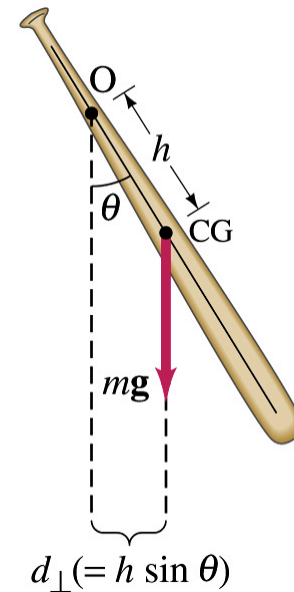
The physical pendulum.

- The period of the physical pendulum is equal to

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{I}{mgh}}$$

- We can double check our answer by requiring that the simple pendulum is a special case of the physical pendulum ($h = L$, $I = mL^2$):

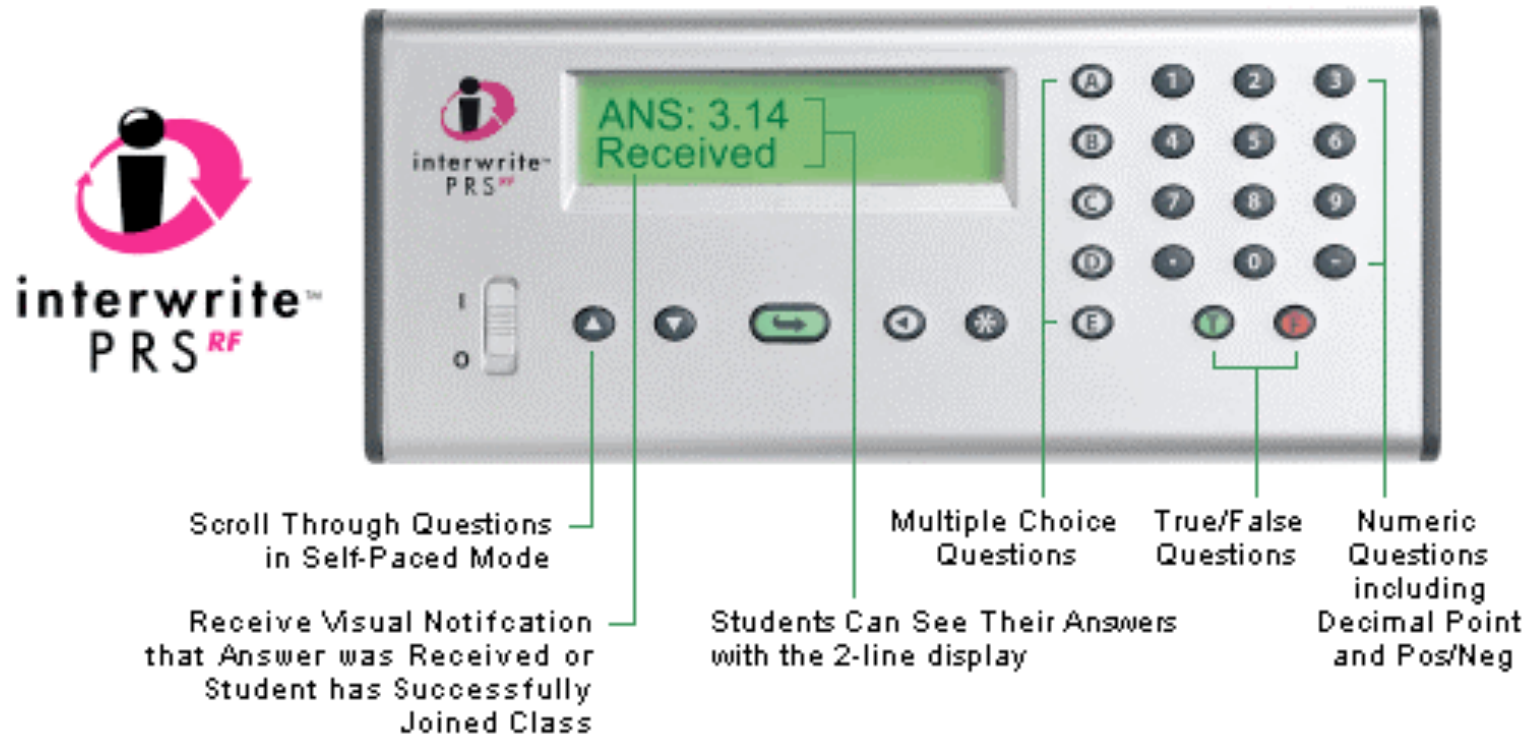
$$T = 2\pi \sqrt{\frac{I}{mgh}} = 2\pi \sqrt{\frac{mL^2}{mgL}} = 2\pi \sqrt{\frac{L}{g}}$$



Physics 121.

Quiz lecture 21.

- The quiz today will have 3 questions!



Simple Harmonic Motion (SHM).

The equation of motion.

- All examples of SHM were derived from the following equation of motion:

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

- The most general solution to the equation is

$$x(t) = A \cos(\omega t + \alpha) + B \sin(\omega t + \beta)$$

Simple Harmonic Motion (SHM).

The equation of motion.

- If $A = B$

$$\begin{aligned}x(t) &= A \cos(\omega t + \alpha) + B \sin(\omega t + \beta) = \\&= A \left(\sin\left(\frac{1}{2}\pi - \omega t - \alpha\right) + \sin(\omega t + \beta) \right) = \\&= 2A \sin\left(\frac{1}{4}\pi + \frac{\beta}{2} - \frac{\alpha}{2}\right) \cos\left(\frac{1}{4}\pi - \omega t - \frac{\beta}{2} - \frac{\alpha}{2}\right)\end{aligned}$$

which is SHM.

Damped Harmonic Motion.

- Consider what happens when in addition to the restoring force a damping force (such as the drag force) is acting on the system:

$$F = -kx - b \frac{dx}{dt}$$

- The equation of motion is now given by:

$$\frac{d^2x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \frac{k}{m} x = 0$$

Damped Harmonic Motion.

- The general solution of this equation of motion is

$$x(t) = Ae^{i\omega t}$$

- If we substitute this solution in the equation of motion we find

$$-\omega^2 Ae^{i\omega t} + i\omega \frac{b}{m} Ae^{i\omega t} + \frac{k}{m} Ae^{i\omega t} = 0$$

- In order to satisfy the equation of motion, the angular frequency must satisfy the following condition:

$$\omega^2 - i\omega \frac{b}{m} - \frac{k}{m} = 0$$

Damped Harmonic Motion.

- We can solve this equation and determine the two possible values of the angular velocity:

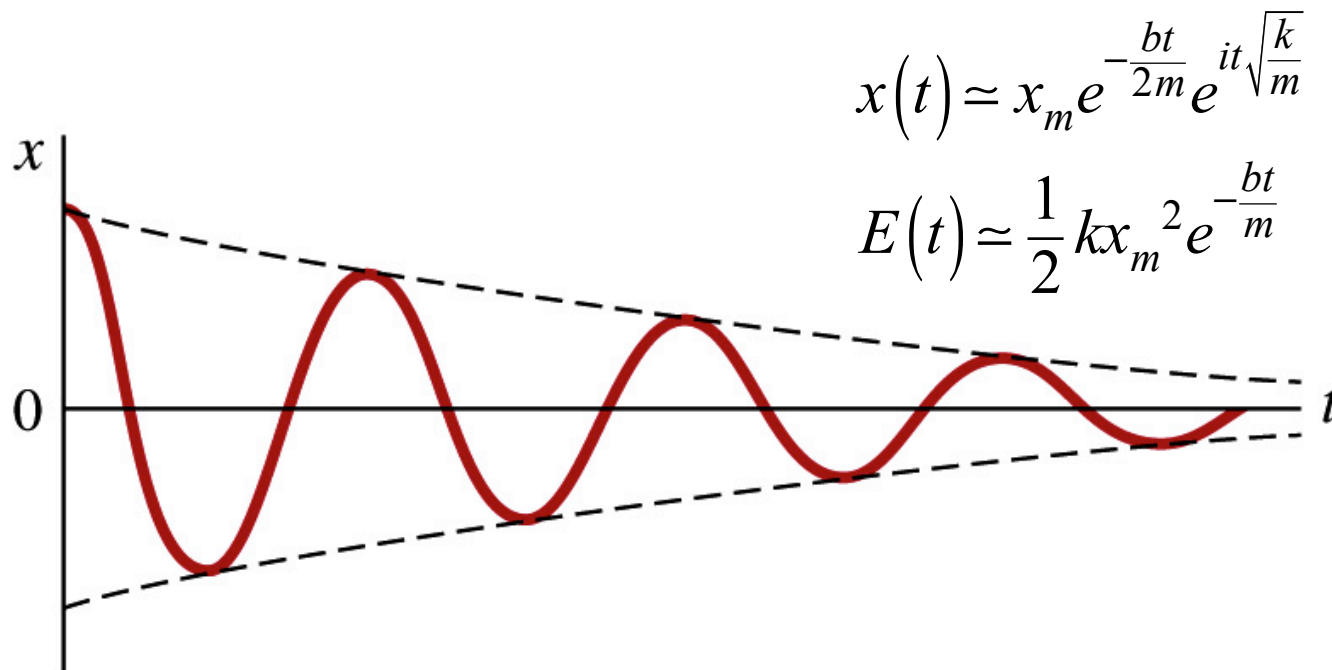
$$\omega = \frac{1}{2} \left(i \frac{b}{m} \pm \sqrt{4 \frac{k}{m} - \frac{b^2}{m^2}} \right) \simeq \frac{1}{2} i \frac{b}{m} \pm \sqrt{\frac{k}{m}}$$

- The solution to the equation of motion is thus given by

$$x(t) \simeq x_m e^{-\frac{bt}{2m}} e^{it\sqrt{\frac{k}{m}}}$$

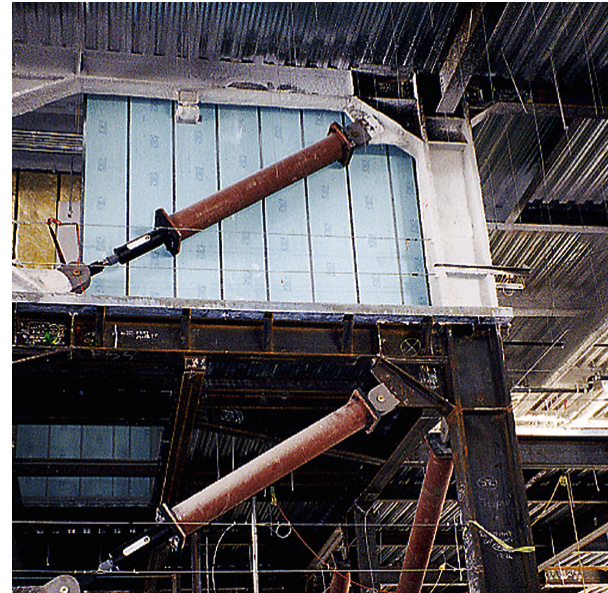
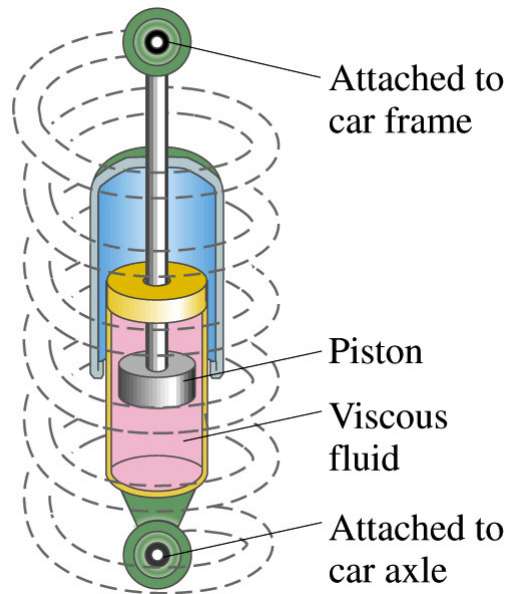
↑ ↑
Damping Term SHM Term

Damped Harmonic Motion.



The general solution contains a SHM term,
with an amplitude that decreases as function of time

Damped Harmonic Motion has many practical applications.



Damping is not always a curse.

Driven Harmonic Motion.

- Consider what happens when we apply a time-dependent force $F(t)$ to a system that normally would carry out SHM with an angular frequency ω_0 .
- Assume the external force $F(t) = mF_0\sin(\omega t)$. The equation of motion can now be written as

$$\frac{d^2x}{dt^2} = -\omega_0^2 x + F_0 \sin(\omega t)$$

- The steady state motion of this system will be harmonic motion with an angular frequency equal to the angular frequency of the driving force.

Driven Harmonic Motion.

- Consider the general solution

$$x(t) = A \cos(\omega t + \phi)$$

- The parameters in this solution must be chosen such that the equation of motion is satisfied. This requires that

$$-\omega^2 A \cos(\omega t + \phi) + \omega_0^2 A \cos(\omega t + \phi) - F_0 \sin(\omega t) = 0$$

- This equation can be rewritten as

$$\begin{aligned} &(\omega_0^2 - \omega^2) A \cos(\omega t) \cos(\phi) - \\ &(\omega_0^2 - \omega^2) A \sin(\omega t) \sin(\phi) - F_0 \sin(\omega t) = 0 \end{aligned}$$

Driven Harmonic Motion.

- Our general solution must thus satisfy the following condition:

$$(\omega_0^2 - \omega^2) A \cos(\omega t) \cos(\phi) - \{(\omega_0^2 - \omega^2) A \sin(\phi) - F_0\} \sin(\omega t) = 0$$

- Since this equation must be satisfied at all time, we must require that the coefficients of $\cos(\omega t)$ and $\sin(\omega t)$ are 0. This requires that

$$(\omega_0^2 - \omega^2) A \cos(\phi) = 0$$

and

$$(\omega_0^2 - \omega^2) A \sin(\phi) - F_0 = 0$$

Driven Harmonic Motion.

- The interesting solutions are solutions where $A \neq 0$ and $\omega \neq \omega_0$. In this case, our general solution can only satisfy the equation of motion if

$$\cos(\phi) = 0$$

and

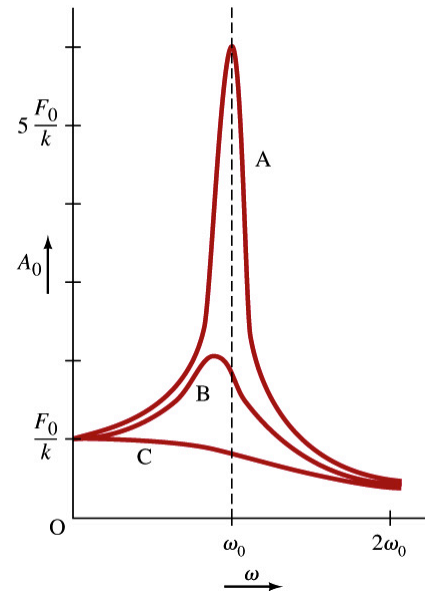
$$(\omega_0^2 - \omega^2) A \sin(\phi) - F_0 = (\omega_0^2 - \omega^2) A - F_0 = 0$$

- The amplitude of the motion is thus equal to

$$A = \frac{F_0}{(\omega_0^2 - \omega^2)}$$

Driven Harmonic Motion.

- If the driving force has a frequency close to the natural frequency of the system, the resulting amplitudes can be very large even for small driving amplitudes. The system is said to be in resonance.
- In realistic systems, there will also be a damping force. Whether or not resonance behavior will be observed will depend on the strength of the damping term.



Driven Harmonic Motion.



Done for today!

Thursday: Temperature and Heat!



Unusually Strong Cyclone Off the Brazilian Coast: A lot of Rotational Motion!

Credit: [Jacques Descloîtres](#), [MODIS Land Rapid Response Team](#), [GSFC](#), [NASA](#)