Quantum Mechanics I - Module 3

1. The ordinary differential equation $\frac{d\psi(x)}{dz} = \left[\psi(x)\right]^2$ has the general solution $\psi(x) = \frac{1}{\alpha - x}$ where α can be any real (or complex) number. Examine the sum of two specific solutions: $\psi_1(x) = \frac{1}{1-x}$ and $\psi_2(x) = \frac{1}{2-x}$

(a) Does the new function $\psi_{1+2}(x) = \frac{1}{1-x} + \frac{1}{2-x}$ satisfy the original differential equation?

(b) From your knowledge of differential equations, specifically the word "linear", what is it about the differential equation that causes solutions to be non-additive?

2. Consider a particle of mass m inside a box of size a with infinite walls:

$$V(x) = \begin{cases} 0 & 0 \le x \le a \\ \infty & \text{elsewhere} \end{cases}$$

We want to determine the time evolution of a wavefunction that is specified at $t = 0$. Assume that $\psi(x, t = 0) = C(2 \sin kx + 3\sin 2kx + \sin 3kx)$, where $k = \pi/a$.

(a). Determine the normalization coefficient C so that $|\psi(x, t)|^2$ can be interpreted as a probability density. The following integral may be useful:

$$\int_0^a \sin(nkx)\sin(mkx)\,dx = \delta_{nm}\frac{a}{2}$$

(b). Expand the wavefunction at the initial time $\psi(x, t = 0)$ in terms of the eigenfunctions $u_n(x)$ of the infinite box, i.e. determine the coefficients

$$c_n = \int_{-\infty}^{\infty} u_n^*(x)\psi(x,0)\,dx$$

so that you can write $\psi(x, 0)$ as a superposition of eigenstates of the infinite box.

(c) Using the known time evolution of eigenstates, find $\psi(x, t)$ at an arbitrary later time t.

(d) Is the motion periodic? In other words, is there a time T with $\psi(x, 0) = \psi(x, T)$?

(e) If a measurement of the particle's energy is performed, what will be the outcome (or outcomes), and with what probability will those values be measured?