Physics 237, Midterm Exam #1

Thursday February 18, 2010

12.30 pm – 1.45 pm

Do not turn the pages of the exam until you are instructed to do so.

Exam rules: You may use *only* a writing instrument while taking this test. You may *not* consult any calculators, computers, books, or each other.

1. Problems 1 and 2 must be answered in booklet # 1.

2. Problems 3 and 4 must be answered in booklet # 2.

3. The answers need to be well motivated and expressed in terms of the variables used in the problem. You will receive partial credit where appropriate, but only when we can read your solution. Answers that are not motivated will not receive any credit, even if correct.

At the end of the exam, you need to hand in your exam, your "cheat sheet", and the two blue exam booklets. All items must be clearly labeled with your name, your student ID number, and the day/time of your workshop.

Name: ____

ID number: ____

Workshop Day/Time: ____________________________________

Problem 1 (35 points) **ANSWER IN BOOKLET 1**

Consider a particle of mass m moving with a linear momentum p. Assume the velocity of the particle is much less than the speed of light and relativistic effects do not need to be considered. In order to describe the particle in terms of a matter wave, we first consider the following matter wave:

$$\Psi(x,t) = \sin\left(2\pi\left(-\frac{x}{\lambda} - \nu t\right)\right) = \sin\left(2\pi(-\kappa x - \nu t)\right)$$

a) What is the propagation velocity of this matter wave? Specify both the magnitude and the direction of the propagation velocity. Express your answer in terms of κ and ν.

b) How does the propagation velocity of the matter wave compare with the velocity of the particle?

Now consider that we describe the particle by the following matter wave:

$$\Psi(x,t) = \sin\left(2\pi(-\kappa x - \nu t)\right) + \sin\left(2\pi\left(-(\kappa + d\kappa)x - (\nu + d\nu)t\right)\right)$$

where $d\kappa << \kappa$ and $d\nu << \nu$.

c) This matter wave has a low- and a high-frequency component. What are the propagation velocities associated with the low- and the high-frequency components? Specify both the magnitude and the direction of these propagation velocities. Express your answers in terms of κ, ν, $d\kappa$, and $d\nu$.

d) How do the propagation velocities of the matter wave obtained in c) compare with the velocity of the particle?

Your answers need to be well motivated. A correct answer without any motivation will not receive any credit.

Problem 2 (30 points) **ANSWER IN BOOKLET 1**

Consider a photon with an energy E_γ travelling in a vacuum. The energy of the photon is larger than 2 times the rest energy of the electron ($E_\gamma > 2\, m_e c^2$).

a) Can the photon convert all of its energy by creating an electron-positron pair? If you answer is yes, calculate the total kinetic energy of the electron and the positron. If your answer is no, show why this process cannot happen in a vacuum.

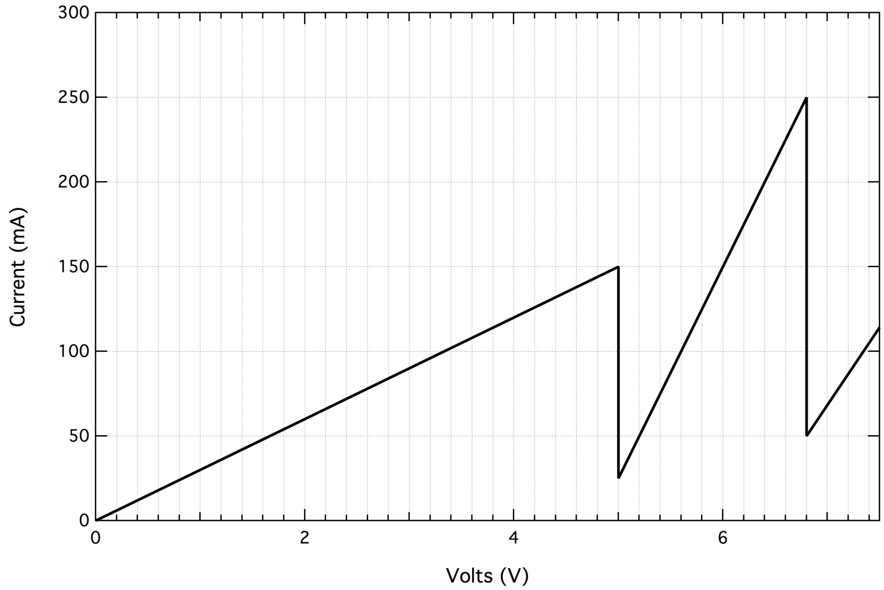
Now consider a photon with an energy E_γ producing an electron-positron pair in the vicinity of a nucleus of mass M. The positron is at rest while the electron has a kinetic energy equal to $2\, m_e c^2$ and moves in the same direction as the pair-producing photon was moving.

b) What was the energy of the pair-producing photon?

c) What fraction of the photon's linear momentum is transferred to the nucleus?

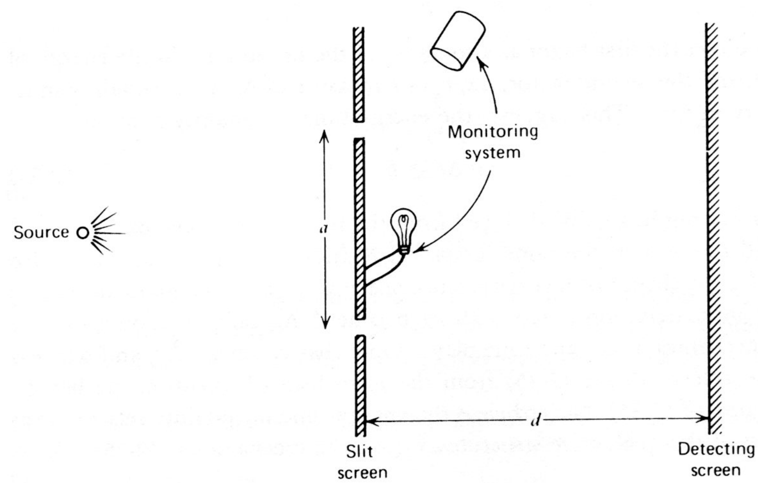
Problem 3 (30 points) **ANSWER IN BOOKLET 2**

The graph shows the voltage dependence of the current you measure in the Franck-Hertz experiment you carry out in the advanced laboratory.



a) Based on the information provided in the Figure, construct an energy-level diagram of the atoms used in the experiment.
b) What are the energies of the photons that are emitted by the atoms used in the experiment when the experiment is operated with an accelerating potential of 7 V?

Consider the two-slit experiment used to observe electron diffraction shown in the Figure. The condition for constructive interference is $\sin\theta = n\lambda / a$. The distance between adjacent maxima on the screen is $d\sin\theta_{n+1} - d\sin\theta_n = d\lambda / a$.



c) After observing the interference pattern we install a monitor system that determines the position of the electron just behind the slit screen with an accuracy $\Delta y < a/2$ so that we can tell through which slit each electron went. Show that this measurement will wipe out the interference pattern.

Problem 4 (5 points) **ANSWER IN BOOKLET 2**

How many times did the Yankees win the world series?

1. 4
2. 25
3. 27
4. 45

APPENDIX

A Mathematical Formulas

A–1 Quadratic Formula

If $$ax^2 + bx + c = 0$$

then $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

A–2 Binomial Expansion

$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)}{2!}x^2 \pm \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

$$(x + y)^n = x^n\left(1 + \frac{y}{x}\right)^n = x^n\left(1 + n\frac{y}{x} + \frac{n(n-1)}{2!}\frac{y^2}{x^2} + \cdots\right)$$

A–3 Other Expansions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$$

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2}{15}\theta^5 + \cdots \qquad |\theta| < \frac{\pi}{2}$$

In general: $$f(x) = f(0) + \left(\frac{df}{dx}\right)_0 x + \left(\frac{d^2f}{dx^2}\right)_0 \frac{x^2}{2!} + \cdots$$

A–4 Exponents

$$(a^n)(a^m) = a^{n+m} \qquad \frac{1}{a^n} = a^{-n}$$

$$(a^n)(b^n) = (ab)^n \qquad a^n a^{-n} = a^0 = 1$$

$$(a^n)^m = a^{nm} \qquad a^{\frac{1}{2}} = \sqrt{a}$$

A–5 Areas and Volumes

Object	Surface area	Volume
Circle, radius r	πr^2	—
Sphere, radius r	$4\pi r^2$	$\frac{4}{3}\pi r^3$
Right circular cylinder, radius r, height h	$2\pi r^2 + 2\pi rh$	$\pi r^2 h$
Right circular cone, radius r, height h	$\pi r^2 + \pi r\sqrt{r^2 + h^2}$	$\frac{1}{3}\pi r^2 h$

A-8 Vectors

Vector addition is covered in Sections 3–2 to 3–5.
Vector multiplication is covered in Sections 3–3, 7–2, and 11–2.

A-9 Trigonometric Functions and Identities

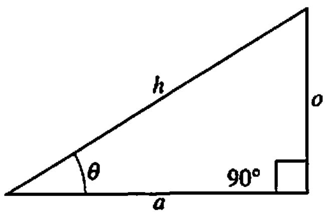


FIGURE A-5

The trigonometric functions are defined as follows (see Fig. A–5, o = side opposite, a = side adjacent, h = hypotenuse. Values are given in Table A–2):

$$\sin\theta = \frac{o}{h} \qquad \csc\theta = \frac{1}{\sin\theta} = \frac{h}{o}$$

$$\cos\theta = \frac{a}{h} \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{h}{a}$$

$$\tan\theta = \frac{o}{a} = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{a}{o}$$

and recall that

$$a^2 + o^2 = h^2 \qquad \text{[Pythagorean theorem].}$$

FIGURE A-6

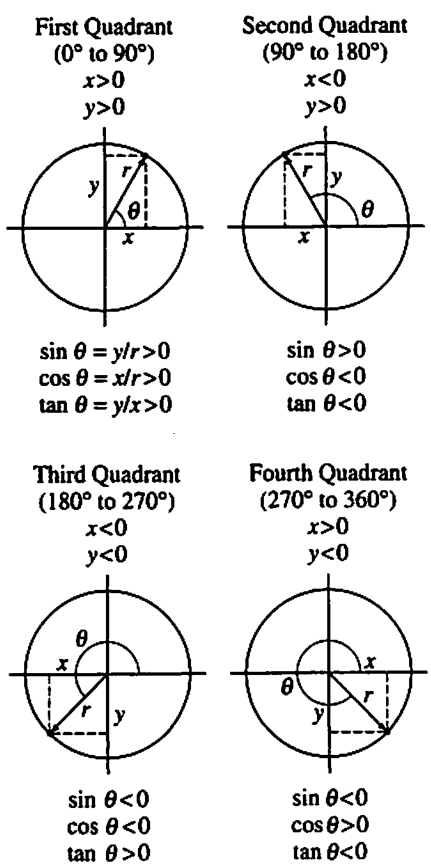


Figure A–6 shows the signs (+ or −) that cosine, sine, and tangent take on for angles θ in the four quadrants (0° to 360°). Note that angles are measured counterclockwise from the x axis as shown; negative angles are measured from *below* the x axis, clockwise: for example, $-30° = +330°$, and so on.

The following are some useful identities among the trigonometric functions:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sec^2\theta - \tan^2\theta = 1, \quad \csc^2\theta - \cot^2\theta = 1$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$

$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$

$$\sin(180° - \theta) = \sin\theta$$

$$\cos(180° - \theta) = -\cos\theta$$

$$\sin(90° - \theta) = \cos\theta$$

$$\cos(90° - \theta) = \sin\theta$$

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = -\tan\theta$$

$$\sin\tfrac{1}{2}\theta = \sqrt{\frac{1-\cos\theta}{2}}, \quad \cos\tfrac{1}{2}\theta = \sqrt{\frac{1+\cos\theta}{2}}, \quad \tan\tfrac{1}{2}\theta = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

$$\sin A \pm \sin B = 2\sin\left(\frac{A \pm B}{2}\right)\cos\left(\frac{A \mp B}{2}\right).$$

FIGURE A-7

For any triangle (see Fig. A–7):

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c} \qquad \text{[Law of sines]}$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma. \qquad \text{[Law of cosines]}$$

Values of sine, cosine, tangent are given in Table A–2.

APPENDIX

B Derivatives and Integrals

Derivatives: General Rules

(See also Section 2–3.)

$$\frac{dx}{dx} = 1$$

$$\frac{d}{dx}[af(x)] = a\frac{df}{dx} \qquad [a = \text{constant}]$$

$$\frac{d}{dx}[f(x) + g(x)] = \frac{df}{dx} + \frac{dg}{dx}$$

$$\frac{d}{dx}[f(x)g(x)] = \frac{df}{dx}g + f\frac{dg}{dx}$$

$$\frac{d}{dx}[f(y)] = \frac{df}{dy}\frac{dy}{dx} \qquad [\text{chain rule}]$$

$$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)} \qquad \text{if } \frac{dy}{dx} \neq 0.$$

Derivatives: Particular Functions

$$\frac{da}{dx} = 0 \qquad [a = \text{constant}]$$

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}\sin ax = a\cos ax$$

$$\frac{d}{dx}\cos ax = -a\sin ax$$

$$\frac{d}{dx}\tan ax = a\sec^2 ax$$

$$\frac{d}{dx}\ln ax = \frac{1}{x}$$

$$\frac{d}{dx}e^{ax} = ae^{ax}$$

Indefinite Integrals: General Rules

(See also Section 7–3.)

$$\int dx = x$$

$$\int af(x)\,dx = a\int f(x)\,dx \qquad [a = \text{constant}]$$

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int u\,dv = uv - \int v\,du \qquad [\text{integration by parts: see also B}$$

Indefinite Integrals: Particular Functions

(An arbitrary constant can be added to the right side of each equation.)

$$\int a\,dx = ax \qquad [a = \text{constant}]$$

$$\int x^m\,dx = \frac{1}{m+1}x^{m+1} \qquad [m \neq -1]$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax$$

$$\int \tan ax\,dx = \frac{1}{a}\ln|\sec ax|$$

$$\int \frac{1}{x}\,dx = \ln x$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left(x + \sqrt{x^2 \pm a^2}\right)$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) = -\cos^{-1}\left(\frac{x}{a}\right) \qquad [\text{if } x^2 \leq a^2]$$

$$\int \frac{dx}{(x^2 \pm a^2)^{\frac{3}{2}}} = \frac{\pm x}{a^2\sqrt{x^2 \pm a^2}}$$

$$\int \frac{x\,dx}{(x^2 \pm a^2)^{\frac{3}{2}}} = \frac{-1}{\sqrt{x^2 \pm a^2}}$$

$$\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$$

$$\int xe^{-ax}\,dx = -\frac{e^{-ax}}{a^2}(ax + 1)$$

$$\int x^2e^{-ax}\,dx = -\frac{e^{-ax}}{a^3}(a^2x^2 + 2ax + 2)$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right) \qquad [x^2 > a^2]$$

$$= -\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) \qquad [x^2 < a^2]$$

A Few Definite Integrals

$$\int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}$$

$$\int_0^\infty e^{-ax^2}\,dx = \sqrt{\frac{\pi}{4a}}$$

$$\int_0^\infty xe^{-ax^2}\,dx = \frac{1}{2a}$$

$$\int_0^\infty x^2e^{-ax^2}\,dx = \sqrt{\frac{\pi}{16a^3}}$$

$$\int_0^\infty x^3e^{-ax^2}\,dx = \frac{1}{2a^2}$$

$$\int_0^\infty x^{2n}e^{-ax^2}\,dx = \frac{1\cdot3\cdot5\cdots(2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}$$

Integration by Parts

Sometimes a difficult integral can be simplified by carefully choosing the functions u and v in the identity:

$$\int u\,dv = uv - \int v\,du. \qquad [\text{Integration by parts}]$$

This identity follows from the property of derivatives

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

or as differentials: $d(uv) = u\,dv + v\,du$.

For example $\int xe^{-x}\,dx$ can be integrated by choosing $u = x$ and $dv = e^{-x}\,dx$ in the "integration by parts" equation above:

$$\int xe^{-x}\,dx = (x)(-e^{-x}) + \int e^{-x}\,dx$$

$$= -xe^{-x} - e^{-x} = -(x+1)e^{-x}.$$

APPENDIX E

*Useful Integrals**

E.1 Algebraic Functions

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right), \qquad \left|\tan^{-1}\left(\frac{x}{a}\right)\right| < \frac{\pi}{2} \tag{E.1}$$

$$\int \frac{x\,dx}{a^2 + x^2} = \frac{1}{2}\ln(a^2 + x^2) \tag{E.2}$$

$$\int \frac{dx}{x(a^2 + x^2)} = \frac{1}{2a^2}\ln\left(\frac{x^2}{a^2 + x^2}\right) \tag{E.3}$$

$$\int \frac{dx}{a^2x^2 - b^2} = \frac{1}{2ab}\ln\left(\frac{ax - b}{ax + b}\right) \tag{E.4a}$$

$$= -\frac{1}{ab}\coth^{-1}\left(\frac{ax}{b}\right), \qquad a^2x^2 > b^2 \tag{E.4b}$$

$$= -\frac{1}{ab}\tanh^{-1}\left(\frac{ax}{b}\right), \qquad a^2x^2 < b^2 \tag{E.4c}$$

$$\int \frac{dx}{\sqrt{a + bx}} = \frac{2}{b}\sqrt{a + bx} \tag{E.5}$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln(x + \sqrt{x^2 + a^2}) \tag{E.6}$$

*This list is confined to those (nontrivial) integrals that arise in the text and in the problems. Extremely useful compilations are, for example, Pierce and Foster (Pi57) and Dwight (Dw61).

$$\int \frac{x^2\,dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \tag{E.7}$$

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}} = \frac{1}{\sqrt{a}}\ln(2\sqrt{a}\sqrt{ax^2 + bx + c} + 2ax + b), \quad a > 0 \tag{E.8a}$$

$$= \frac{1}{\sqrt{a}}\sinh^{-1}\left(\frac{2ax + b}{\sqrt{4ac - b^2}}\right), \quad \begin{cases} a > 0 \\ 4ac > b^2 \end{cases} \tag{E.8b}$$

$$= -\frac{1}{\sqrt{-a}}\sin^{-1}\left(\frac{2ax + b}{\sqrt{b^2 - 4ac}}\right), \quad \begin{cases} a < 0 \\ b^2 > 4ac \\ |2ax + b| < \sqrt{b^2 - 4ac} \end{cases} \tag{E.8c}$$

$$\int \frac{x\,dx}{\sqrt{ax^2 + bx + c}} = \frac{1}{a}\sqrt{ax^2 + bx + c} - \frac{b}{2a}\int \frac{dx}{\sqrt{ax^2 + bx + c}} \tag{E.9}$$

$$\int \frac{dx}{x\sqrt{ax^2 + bx + c}} = -\frac{1}{\sqrt{c}}\sinh^{-1}\left(\frac{bx + 2c}{|x|\sqrt{4ac - b^2}}\right), \quad \begin{cases} c > 0 \\ 4ac > b^2 \end{cases} \tag{E.10a}$$

$$= \frac{1}{\sqrt{-c}}\sin^{-1}\left(\frac{bx + 2c}{|x|\sqrt{b^2 - 4ac}}\right), \quad \begin{cases} c < 0 \\ b^2 > 4ac \end{cases} \tag{E.10b}$$

$$= -\frac{1}{\sqrt{c}}\ln\left(\frac{2\sqrt{c}}{x}\sqrt{ax^2 + bx + c} + \frac{2c}{x} + b\right), \quad c > 0 \tag{E.10c}$$

$$\int \sqrt{ax^2 + bx + c}\,dx = \frac{2ax + b}{4a}\sqrt{ax^2 + bx + c} + \frac{4ac - b^2}{8a}\int \frac{dx}{\sqrt{ax^2 + bx + c}} \tag{E.11}$$

E.2 Trigonometric Functions

$$\int \sin^2 x\,dx = \frac{x}{2} - \frac{1}{4}\sin 2x \tag{E.12}$$

$$\int \cos^2 x\,dx = \frac{x}{2} + \frac{1}{4}\sin 2x \tag{E.13}$$

$$\int \frac{dx}{a + b\sin x} = \frac{2}{\sqrt{a^2 - b^2}}\tan^{-1}\left[\frac{a\tan(x/2) + b}{\sqrt{a^2 - b^2}}\right], \quad a^2 > b^2 \tag{E.14}$$

$$\int \frac{dx}{a + b\cos x} = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1}\left[\frac{(a-b)\tan(x/2)}{\sqrt{a^2 - b^2}}\right], \quad a^2 > b^2 \qquad \text{(E.15)}$$

$$\int \frac{dx}{(a + b\cos x)^2} = \frac{b\sin x}{(b^2 - a^2)(a + b\cos x)} - \frac{a}{b^2 - a^2}\int \frac{dx}{a + b\cos x} \qquad \text{(E.16)}$$

$$\int \tan x\, dx = -\ln|\cos x| \qquad \text{(E.17a)}$$

$$\int \tanh x\, dx = \ln \cosh x \qquad \text{(E.17b)}$$

$$\int e^{ax}\sin x\, dx = \frac{e^{ax}}{a^2 + 1}(a\sin x - \cos x) \qquad \text{(E.18a)}$$

$$\int e^{ax}\sin^2 x\, dx = \frac{e^{ax}}{a^2 + 4}\left(a\sin^2 x - 2\sin x\cos x + \frac{2}{a}\right) \qquad \text{(E.18b)}$$

$$\int_{-\infty}^{\infty} e^{-ax^2}\, dx = \sqrt{\pi/a} \qquad \text{(E.18c)}$$

E.3 Gamma Functions

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}\, dx \qquad \text{(E.19a)}$$

$$= \int_0^1 [\ln(1/x)]^{n-1}\, dx \qquad \text{(E.19b)}$$

$$\Gamma(n) = (n-1)!, \quad \text{for } n = \text{positive integer} \qquad \text{(E.19c)}$$

$$n\Gamma(n) = \Gamma(n+1) \qquad \text{(E.20)}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \qquad \text{(E.21)}$$

$$\Gamma(1) = 1 \qquad \text{(E.22)}$$

$$\Gamma\left(1\frac{1}{4}\right) = 0.906 \qquad \text{(E.23)}$$

$$\Gamma\left(1\frac{3}{4}\right) = 0.919 \qquad \text{(E.24)}$$

$$\Gamma(2) = 1 \tag{E.25}$$

$$\int_0^1 \frac{dx}{\sqrt{1 - x^n}} = \frac{\sqrt{\pi}}{n} \frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n} + \frac{1}{2}\right)} \tag{E.26}$$

$$\int_0^1 x^m (1 - x^2)^n \, dx = \frac{\Gamma(n + 1)\Gamma\left(\frac{m + 1}{2}\right)}{2\Gamma\left(n + \frac{m + 3}{2}\right)} \tag{E.27a}$$

$$\int_0^{\pi/2} \cos^n x \, dx = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{n + 1}{2}\right)}{\Gamma\left(\frac{n}{2} + 1\right)}, \quad n > -1 \tag{E.27b}$$

APPENDIX F

Differential Relations in Different Coordinate Systems

F.1 Rectangular Coordinates

$$\text{grad}\, U = \nabla U = \sum_i \mathbf{e}_i \frac{\partial U}{\partial x_i} \tag{F.1}$$

$$\text{div}\, \mathbf{A} = \nabla \cdot \mathbf{A} = \sum_i \frac{\partial A_i}{\partial x_i} \tag{F.2}$$

$$\text{curl}\, \mathbf{A} = \nabla \times \mathbf{A} = \sum_{i,j,k} \varepsilon_{ijk} \frac{\partial A_k}{\partial x_j} \mathbf{e}_i \tag{F.3}$$

$$\nabla^2 U = \nabla \cdot \nabla U = \sum_i \frac{\partial^2 U}{\partial x_i^2} \tag{F.4}$$

F.2 Cylindrical Coordinates

Refer to Figures F-1 and F-2.

$$x_1 = r\cos\phi, \quad x_2 = r\sin\phi, \quad x_3 = z \tag{F.5}$$

$$r = \sqrt{x_1^2 + x_2^2}, \quad \phi = \tan^{-1}\frac{x_2}{x_1}, \quad z = x_3 \tag{F.6}$$

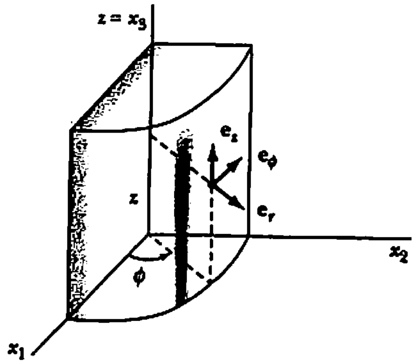


FIGURE F-1

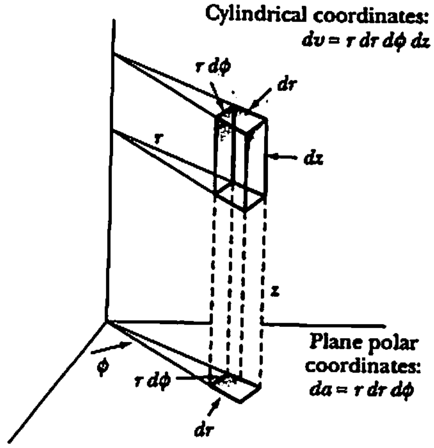


FIGURE F-2

$$ds^2 = dr^2 + r^2\,d\phi^2 + dz^2 \tag{F.7}$$

$$dv = r\,dr\,d\phi\,dz \tag{F.8}$$

$$\text{grad}\ \psi = \nabla\psi = \mathbf{e}_r\frac{\partial\psi}{\partial r} + \mathbf{e}_\phi\frac{1}{r}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_z\frac{\partial\psi}{\partial z} \tag{F.9}$$

$$\text{div}\ \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z} \tag{F.10}$$

$$\text{curl}\ \mathbf{A} = \mathbf{e}_r\left(\frac{1}{r}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right) + \mathbf{e}_\phi\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \mathbf{e}_z\left(\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi) - \frac{1}{r}\frac{\partial A_r}{\partial\phi}\right) \tag{F.11}$$

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} \tag{F.12}$$

F.3 Spherical Coordinates

Refer to Figures F-3 and F-4

$$x_1 = r \sin\theta \cos\phi, \quad x_2 = r \sin\theta \sin\phi, \quad x_3 = r \cos\theta \tag{F.13}$$

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad \theta = \cos^{-1}\frac{x_3}{r}, \quad \phi = \tan^{-1}\frac{x_2}{x_1} \tag{F.14}$$

$$ds^2 = dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \tag{F.15}$$

$$dv = r^2 \sin\theta\, dr\, d\theta\, d\phi \tag{F.16}$$

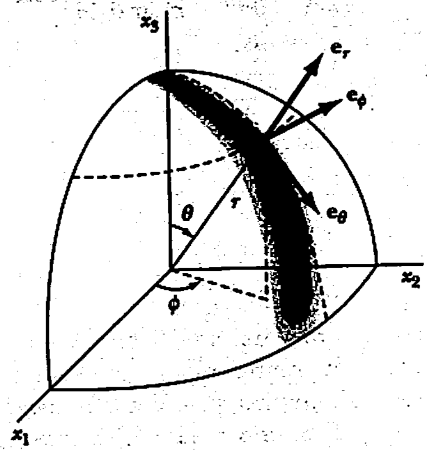


FIGURE F-3

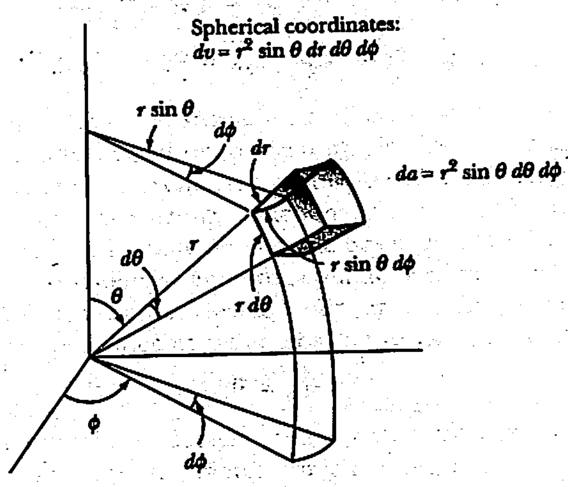


FIGURE F-4

$$\operatorname{grad}\psi = \nabla\psi = \mathbf{e}_r\frac{\partial\psi}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_\phi\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi} \tag{F.17}$$

$$\operatorname{div}\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta} + \frac{\partial A_\phi}{\partial\phi} \tag{F.18}$$

$$\begin{aligned}\operatorname{curl}\mathbf{A} = {} & \mathbf{e}_r\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right] \\ & + \mathbf{e}_\theta\frac{1}{r\sin\theta}\left[\frac{\partial A_r}{\partial\phi} - \sin\theta\frac{\partial}{\partial r}(rA_\phi)\right] + \mathbf{e}_\phi\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\end{aligned} \tag{F.19}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} \tag{F.20}$$

Appendix M

THE LAPLACIAN AND ANGULAR MOMENTUM OPERATORS IN SPHERICAL POLAR COORDINATES

THE LAPLACIAN OPERATOR

The Laplacian operator ∇^2, which enters into the three-dimensional Schroedinger equation, is defined in rectangular coordinates as

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{M-1}$$

We show here how to transform the operator into the form it assumes in spherical polar coordinates, which is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{dr}\right) + \frac{1}{r^2 \sin^2 \theta}\frac{\partial^2}{\partial \varphi^2} + \frac{1}{r^2 \sin \theta}\frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial}{\partial \theta}\right) \tag{M-2}$$

The most straightforward way to carry out the transformation is to make repeated applications of the "chain rule" of partial differentiation. This is a tedious procedure. But the first term of (M-2) can be obtained, without too much tedium, by considering a case in which the Laplacian operates on a function $\psi = \psi(r)$ of the radial coordinate alone. In this case, the derivatives in the last two terms of (M-2) yield zero, and we have

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right)$$

We shall obtain this expression from the expression

$$\nabla^2\psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}$$

which is the Laplacian in rectangular coordinates of (M-1), operating on $\psi(r)$. To do this, we use the relation

$$r = (x^2 + y^2 + z^2)^{1/2}$$

connecting the rectangular and the spherical polar coordinates (see Figure 7-2).

We evaluate

$$\frac{\partial \psi}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial \psi}{\partial r} = \frac{x}{(x^2 + y^2 + z^2)^{1/2}}\frac{\partial \psi}{\partial r} = \frac{x}{r}\frac{\partial \psi}{\partial r}$$

and

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{x}{r}\frac{\partial\psi}{\partial r}\right) = \frac{\partial x}{\partial x}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) + x\frac{\partial}{\partial x}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)$$

$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{r}\frac{\partial\psi}{\partial r} + x\frac{\partial r}{\partial x}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)$$

$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{x^2}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)$$

Similarly, the y and z derivatives yield

$$\frac{\partial^2\psi}{\partial y^2} = \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{y^2}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)$$

and

$$\frac{\partial^2\psi}{\partial z^2} = \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{z^2}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)$$

Adding these three expressions, we obtain

$$\nabla^2\psi = \frac{3}{r}\frac{\partial\psi}{\partial r} + \frac{(x^2+y^2+z^2)}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)$$

or

$$\nabla^2\psi = \frac{3}{r}\frac{\partial\psi}{\partial r} + r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)$$

Now note that the expression we have obtained expands to

$$\nabla^2\psi = \frac{3}{r}\frac{\partial\psi}{\partial r} + r\left(-\frac{1}{r^2}\frac{\partial\psi}{\partial r} + \frac{1}{r}\frac{\partial^2\psi}{\partial r^2}\right)$$

or

$$\nabla^2\psi = \frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial r^2}$$

Also note that the first term of (M-2), that is

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)$$

expands to

$$\nabla^2\psi = \frac{1}{r^2}\left(2r\frac{\partial\psi}{\partial r} + r^2\frac{\partial^2\psi}{\partial r^2}\right)$$

or

$$\nabla^2\psi = \frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial r^2}$$

Comparison shows that the expression we have obtained is identical to the first term of (M-2). The second and third terms can be obtained by taking $\psi = \psi(\varphi)$, and then taking $\psi = \psi(\theta)$.

THE ANGULAR MOMENTUM OPERATORS

In rectangular coordinates, the operators for the three components of orbital angular momentum are

$$L_{x_{op}} = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$
$$L_{y_{op}} = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \qquad \text{(M-3)}$$
$$L_{z_{op}} = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

When transformed to spherical polar coordinates, these operators asssume the forms

$$L_{x_{op}} = i\hbar\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)$$

$$L_{y_{op}} = i\hbar\left(-\cos\varphi\frac{\partial}{\partial\theta} + \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right) \qquad \text{(M-4)}$$

$$L_{z_{op}} = -i\hbar\frac{\partial}{\partial\varphi}$$

We shall show that these are equivalent, taking $L_{z_{op}}$ as the simplest example. To do this, we must use the relations

$$x = r\sin\theta\cos\varphi$$

$$y = r\sin\theta\sin\varphi \qquad \text{(M-5)}$$

$$z = r\cos\theta$$

connecting the rectangular and spherical polar coordinates (see Figure 7-2).

It is easiest if we start by applying the chain rule to $\partial\psi/\partial\varphi$, and obtain

$$\frac{\partial\psi}{\partial\varphi} = \frac{\partial\psi}{\partial x}\frac{\partial x}{\partial\varphi} + \frac{\partial\psi}{\partial y}\frac{\partial y}{\partial\varphi} + \frac{\partial\psi}{\partial z}\frac{\partial z}{\partial\varphi}$$

From (M-5), we have

$$\frac{\partial x}{\partial\varphi} = -r\sin\theta\sin\varphi = -y$$

$$\frac{\partial y}{\partial\varphi} = r\sin\theta\cos\varphi = x$$

$$\frac{\partial z}{\partial\varphi} = 0$$

Thus

$$\frac{\partial\psi}{\partial\varphi} = -y\frac{\partial\psi}{\partial x} + x\frac{\partial\psi}{\partial y}$$

As an operator equation, this reads

$$\frac{\partial}{\partial\varphi} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

which verifies the equivalence of the two forms of $L_{z_{op}}$ quoted in (M-3) and (M-4). Similar calculations will do the same for $L_{x_{op}}$ and $L_{y_{op}}$.

In rectangular coordinates, the operator for the square of the magnitude of the orbital angular momentum is

$$L_{op}^2 = L_{x_{op}}^2 + L_{y_{op}}^2 + L_{z_{op}}^2 \qquad \text{(M-6)}$$

By squaring $L_{x_{op}}$, $L_{y_{op}}$, and $L_{z_{op}}$, and adding, it is found after some manipulation of the sinusoidal functions that

$$L_{op}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right] \qquad \text{(M-7)}$$

Note the relation between (M-7) and the last two terms in (M-2). It forms the basis of an alternative way of obtaining those terms, which can be found in mathematical reference books.

PROBLEM

1. By using the techniques of Appendix M, show that $L_{x_{op}}$ has the form stated in (7-37).