Useful Relations

$$\overline{A} \bullet \overline{B} = A_x B_x + A_y B_y + A_z B_z$$

$$\overline{C} = \overline{A} \times \overline{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\overline{A} \bullet \left(\overline{B} \times \overline{C}\right) = \overline{B} \bullet \left(\overline{C} \times \overline{A}\right) = \overline{C} \bullet \left(\overline{A} \times \overline{B}\right)$$

$$\overline{A} \times \left(\overline{B} \times \overline{C}\right) = \left(\overline{A} \bullet \overline{C}\right)\overline{B} - \left(\overline{A} \bullet \overline{B}\right)\overline{C}$$

$$\left(\overline{A} \times \overline{B}\right) \bullet \left(\overline{C} \times \overline{D}\right) = \overline{A} \bullet \left[\overline{B} \times \left(\overline{C} \times \overline{D}\right)\right]$$

$$\left(\overline{A} \times \overline{B}\right) \times \left(\overline{C} \times \overline{D}\right) = \left[\left(\overline{A} \times \overline{B}\right) \bullet \overline{D}\right]\overline{C} - \left[\left(\overline{A} \times \overline{B}\right) \bullet \overline{C}\right]\overline{D}$$

One-Electron Atoms – Details

The following table lists the $n = 1$, $n = 2$, and $n = 3$ wavefunctions of the one-electron atom.

Table 7-2 Some Eigenfunctions for the One-Electron Atom

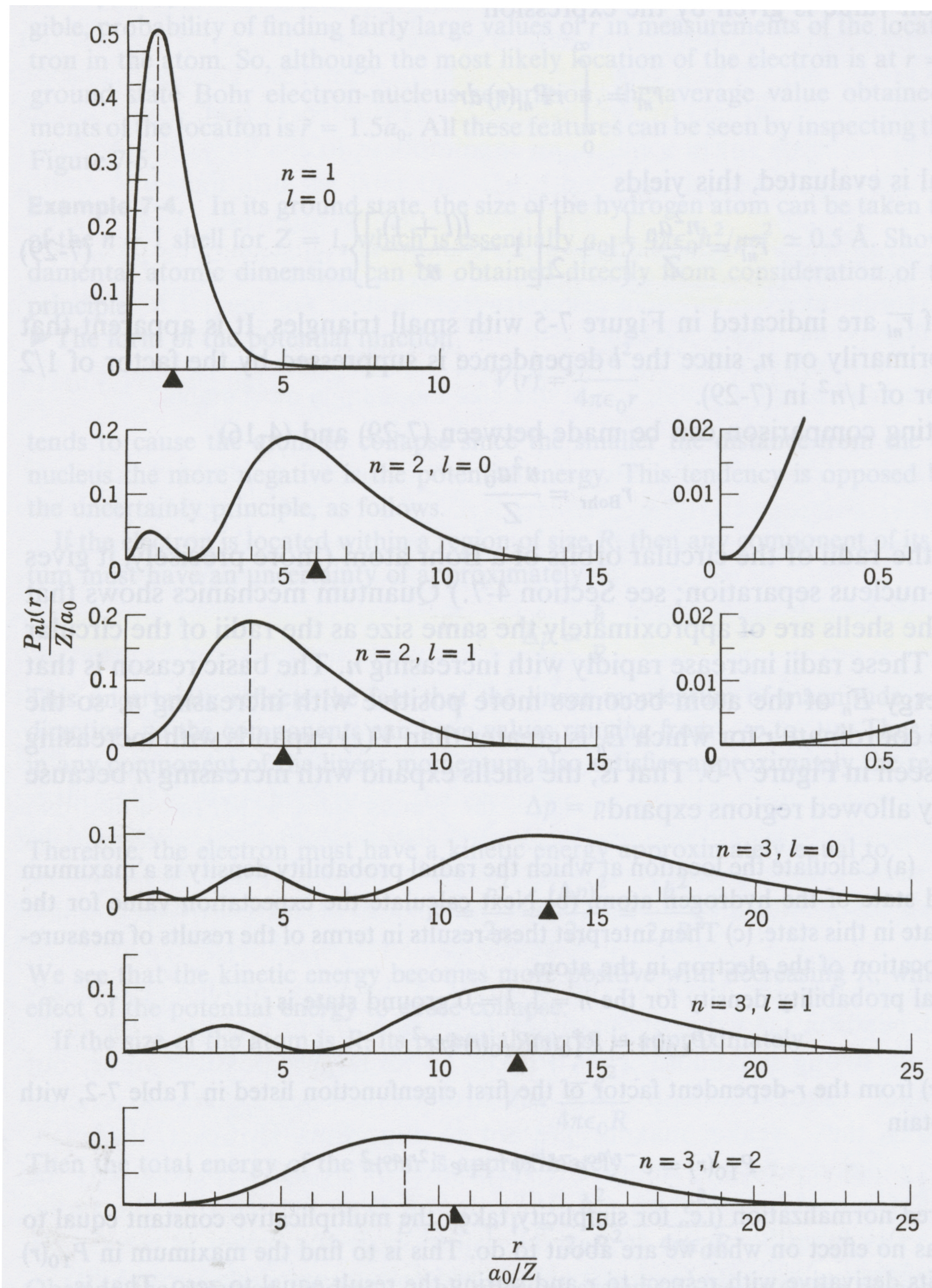
Quantum Numbers n	l	m_l	Eigenfunctions
1	0	0	$\psi_{100} = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0}$
2	0	0	$\psi_{200} = \frac{1}{4\sqrt{2\pi}}\left(\frac{Z}{a_0}\right)^{3/2}\left(2 - \frac{Zr}{a_0}\right) e^{-Zr/2a_0}$
2	1	0	$\psi_{210} = \frac{1}{4\sqrt{2\pi}}\left(\frac{Z}{a_0}\right)^{3/2} \frac{Zr}{a_0} e^{-Zr/2a_0} \cos\theta$
2	1	± 1	$\psi_{21\pm1} = \frac{1}{8\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} \frac{Zr}{a_0} e^{-Zr/2a_0} \sin\theta\, e^{\pm i\varphi}$
3	0	0	$\psi_{300} = \frac{1}{81\sqrt{3\pi}}\left(\frac{Z}{a_0}\right)^{3/2}\left(27 - 18\frac{Zr}{a_0} + 2\frac{Z^2r^2}{a_0^2}\right) e^{-Zr/3a_0}$
3	1	0	$\psi_{310} = \frac{\sqrt{2}}{81\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2}\left(6 - \frac{Zr}{a_0}\right)\frac{Zr}{a_0} e^{-Zr/3a_0} \cos\theta$
3	1	± 1	$\psi_{31\pm1} = \frac{1}{81\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2}\left(6 - \frac{Zr}{a_0}\right)\frac{Zr}{a_0} e^{-Zr/3a_0} \sin\theta\, e^{\pm i\varphi}$
3	2	0	$\psi_{320} = \frac{1}{81\sqrt{6\pi}}\left(\frac{Z}{a_0}\right)^{3/2} \frac{Z^2r^2}{a_0^2} e^{-Zr/3a_0}(3\cos^2\theta - 1)$
3	2	± 1	$\psi_{32\pm1} = \frac{1}{81\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} \frac{Z^2r^2}{a_0^2} e^{-Zr/3a_0} \sin\theta\cos\theta\, e^{\pm i\varphi}$
3	2	± 2	$\psi_{32\pm2} = \frac{1}{162\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} \frac{Z^2r^2}{a_0^2} e^{-Zr/3a_0} \sin^2\theta\, e^{\pm 2i\varphi}$

In these wavefunctions, the parameter a_0 is defined as

$$a_0 = \frac{4\pi\varepsilon_0\hbar^2}{\mu e^2}$$

The energy of each wavefunction is equal to

$$E_n = -\frac{\mu Z^2 e^4}{\left(4\pi\varepsilon_0\right)^2 2\hbar^2 n^2}$$


$n = 1$
$l = 0$
$n = 2, l = 0$
$n = 2, l = 1$
$n = 3, l = 0$
$n = 3, l = 1$
$n = 3, l = 2$
$\frac{P_{nl}(r)}{Z/a_0}$
$\frac{r}{a_0/Z}$
0.5
0.4
0.3
0.2
0.1
0
0.02
0.01
5
10
15
20
25
0.5

Planck's quantization rule:

$$\varepsilon = nh\nu \qquad n = 0,1,2,3,...$$

Stephan's law:

$$R_T = \sigma T^4$$

Rayleigh-Jeans formula for blackbody radiation:

$$\rho_T(\nu)d\nu = \frac{N(\nu)d\nu}{V}kT = \frac{8\pi\nu^2 kT}{c^3}d\nu$$

Planck's blackbody formula:

$$\rho_T(\lambda)d\lambda = \frac{8\pi hc}{\lambda^5}\frac{d\lambda}{e^{hc/\lambda kT}-1}$$

Relativistic total energy:

$$E = \sqrt{(pc)^2 + (mc^2)^2}$$

Compton scattering:

$$\lambda_f - \lambda_i = \frac{h}{mc}(1-\cos\theta) = \lambda_C(1-\cos\theta)$$

$$E_f = \frac{E_i}{1+\frac{E_i}{mc^2}(1-\cos\theta)}$$

Photon intensity as function of thickness:

$$I(t) = I(0)e^{-\sigma\rho t}$$

Bragg relation:

$$2d\sin(\varphi) = n\lambda$$

Electromagnetic wave:

$$\varepsilon(x,t) = A\sin\left\{2\pi\left(\frac{x}{\lambda} - vt\right)\right\}$$

The intensity of an electromagnetic wave is proportional to the average of ε^2 over one cycle:

$$I = \frac{1}{\mu_0 c}\overline{\varepsilon^2}$$

de Broglie relation:

$$\lambda = \frac{h}{p}$$

de Broglie wave

$$\Psi(x,t) = A\sin\left\{2\pi\left(\frac{x}{\lambda} - vt\right)\right\}$$

Heisenberg uncertainty principle:

$$\Delta p_x \Delta x \geq \frac{\hbar}{2}$$

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

Node velocity of the high-frequency component:

$$\text{node velocity} = \frac{dx}{dt} = \frac{\nu}{\kappa} = \omega$$

Node velocity of the envelope (group velocity):

$$\text{node velocity} = \frac{dx}{dt} = \frac{d\nu}{d\kappa}$$

Rutherford scattering:

$$\frac{d\sigma}{d\Omega} = \left(\frac{1}{4\pi\varepsilon_0}\right)^2 \left(\frac{Z_1 Z_2}{E}\right)^2 \frac{1}{\sin^4\left(\frac{\theta}{2}\right)}$$

Energy of an electron in an atom:

$$E = -\left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{\mu\left(Ze^2\right)^2}{2n^2\hbar^2}\left\{1 + \frac{\alpha^2 Z^2}{n}\left\{\frac{1}{n_\theta} - \frac{3}{4n}\right\}\right\}$$

Wilson-Sommerfeld quantization rule:

$$\oint_{\text{One period}} p_q \, dq = n_q h$$

Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi = i\hbar\frac{\partial \Psi}{\partial t}$$

Schrödinger equations for time-independent potentials:

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x)}{\partial x^2} + V(x)\psi(x) = E\psi(x)$$

$$i\hbar\frac{\partial \varphi(t)}{\partial t} = E\varphi(t)$$

Probability distribution:

$$P(x,t) = \Psi^*\Psi$$

Expectation values:

$$\langle x \rangle = \langle \Psi | x | \Psi \rangle = \int_{-\infty}^{\infty} \Psi^* x \Psi \, dx$$

$$\langle p \rangle = \left\langle -i\hbar\frac{\partial}{\partial x} \right\rangle$$

$$\langle E \rangle = \left\langle i\hbar \frac{\partial}{\partial t} \right\rangle$$

Uncertainty in operator O:

$$\Delta O = \sqrt{\langle O^2 \rangle - \langle O \rangle^2}$$

Solutions of the time-independent Schrödinger equation for a free particle ($E > V$):

$$\psi(x) = Ae^{ikx} + Be^{-ikx}$$

Solutions of the time-independent Schrödinger equation for a free particle ($E < V$):

$$\psi(x) = Ce^{\kappa x} + De^{-\kappa x}$$

Reflection coefficient:

$$R = \frac{B^* B}{A^* A}$$

Transmission probability through a Coulomb barrier V:

$$T \simeq e^{-2\int_R^b \sqrt{\frac{2m}{\hbar^2}(V(r)-E)}\,dr} = e^{-2\sqrt{\frac{2m}{\hbar^2}}\int_R^b \sqrt{\left(\frac{1}{4\pi\varepsilon_0}\frac{Z_1 Z_2 e^2}{r} - E\right)}\,dr}$$

Three-dimensional Schrödinger equation for a single-particle system:

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi + V\psi = E\psi$$

where

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

Solution of the three-dimensional Schrödinger equation:

$$\psi_{nlm}(r,\theta,\varphi) = R_{nl}(r)\Theta_{lm}(\theta)\Phi_m(\varphi)$$

Wavefunction for a single-electron atom:

$$\psi_{nlm}(r,\theta,\varphi)=R_{nl}(r)\Theta_{lm}(\theta)\Phi_m(\varphi)$$

Energy without the spin-orbit interaction and relativistic interactions considered:

$$E=-\frac{\mu Z^2 e^4}{2(4\pi\varepsilon_0)^2\hbar^2 n^2}$$

Energy with the spin-orbit interaction and relativistic interactions considered:

$$E=-\frac{\mu Z^2 e^4}{2(4\pi\varepsilon_0)^2\hbar^2 n^2}\left[1+\frac{\alpha^2 Z^2}{n}\left(\frac{1}{j+\frac{1}{2}}-\frac{3}{4n}\right)\right]$$

Angular momentum operators:

$$L_x=-i\hbar\left\{\sin\varphi\frac{\partial}{\partial\theta}+\cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right\}$$

$$L_y=-i\hbar\left\{-\cos\varphi\frac{\partial}{\partial\theta}+\cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right\}$$

$$L_z=-i\hbar\frac{\partial}{\partial\varphi}$$

$$L^2=-\hbar^2\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\}$$

$$L_+=L_x+iL_y$$

$$L_-=L_x-iL_y$$

$$L^2\psi=\ell(\ell+1)\hbar^2\psi$$

$$L_z\psi=m\hbar\psi$$

$$\left[L_x, L_y\right] = i\hbar L_z$$

$$\left[L_y, L_z\right] = i\hbar L_x$$

$$\left[L_z, L_x\right] = i\hbar L_y$$

$$\left[L_+, L_z\right] = -\hbar L_+$$

$$\left[L_-, L_z\right] = +\hbar L_-$$

Dipole moment of electron:

$$\vec{\mu} = -\frac{g_\ell \mu_b}{\hbar}\vec{L}$$

where

$$\mu_b = \frac{e\hbar}{2m} \quad (\text{Bohr magneton})$$

Torque on a dipole in a magnetic field:

$$\vec{\tau} = \vec{\mu} \times \vec{B}$$

Potential energy of a dipole in a magnetic field:

$$U = -\vec{\mu} \bullet \vec{B}$$

Energy shift of the electron in a single-electron atom due to the magnetic field generated by the proton:

$$\Delta E = \frac{1}{2m^2c^2}\frac{1}{r}\frac{dV}{dr}\left(\vec{S} \bullet \vec{L}\right)$$

Total angular momentum:

$$\vec{J} = \vec{L} + \vec{S}$$

The total angular momentum is quantized:

$$J = \sqrt{j(j+1)}\hbar$$

and

$$J_z = m_j\hbar \quad \text{where } m_j = -j, -j+1, \ldots, j-1, j$$

The rate of photon emission:

$$R = \frac{4\pi^3\nu^3}{3\varepsilon_0\hbar c^3}p^2$$

where p is the dipole moment, defined as:

$$\vec{p} = -e\vec{r}$$

Transition rules for one-electron atoms:

$$\Delta m_\ell = 0, \pm 1$$

$$\Delta l = \pm 1$$

$$\Delta j = 0, \pm 1$$

Slater determinant (used to construct asymmetric wavefunctions);

$$\psi_A = \frac{1}{\sqrt{N!}}\begin{vmatrix} \psi_\alpha(1) & \psi_\alpha(2) & . & . \\ \psi_\beta(1) & \psi_\beta(2) & & \\ . & & & \\ . & & & \end{vmatrix}$$

Spin singlet state:

$$\psi_{A,spin} = \frac{1}{2}\sqrt{2}\left\{\left(+\frac{1}{2}, -\frac{1}{2}\right) - \left(-\frac{1}{2}, +\frac{1}{2}\right)\right\}$$

Spin triplet states:

$$\psi_{S,spin} = \begin{cases} \left(+\frac{1}{2},+\frac{1}{2}\right) \\ \frac{1}{2}\sqrt{2}\left\{\left(+\frac{1}{2},-\frac{1}{2}\right)+\left(-\frac{1}{2},+\frac{1}{2}\right)\right\} \\ \left(-\frac{1}{2},-\frac{1}{2}\right) \end{cases}$$

Multi-electron wavefunction:

$$\psi_{multi-electron} = \psi_{spatial}\psi_{spin}$$

The modified potential for a multi-electron atom:

$$V_n(r) = -\frac{1}{4\pi\varepsilon_0}\frac{Z_n e^2}{r}$$

- For the innermost shell (n = 1): $Z_n = Z - 2$.
- For the next shell (n = 2): $Z_n = Z - 10$.
- For the outermost shell: $Z_n = n$.

Energy of the innermost electron in a multi-electron atom:

$$E_1 = (Z-2)^2 E_{1,H}$$

Energy of the outermost electron in a multi-electron atom:

$$E_n = E_{1,H}$$

Average radius of electrons in the outermost shell of a multi-electron atom:

$$\bar{r} = na_0$$

In terms of increasing energy, the sub-shells of multi-elecron atoms are arranged in the following way:

$$1s, 2s, 2p, 3s, 3p, 4s, 3d, \ldots.$$

Fine structure in Alkali atoms:

$$\Delta E = \frac{\hbar^2}{2m^2c^2}\left\{j\left(j+1\right)-\ell\left(\ell+1\right)-s\left(s+1\right)\right\}\frac{1}{r}\frac{dV}{dr}$$

Lande interval rule:

$$\varepsilon_{j+1\to j} = 2k\left(j+1\right)$$

Transition rules for optical electrons:

1. Transitions involve the change of n and ℓ number of one electron. Transitions between states that require the change in quantum numbers of more than one electron are extremely unlikely to be observed.

2. The change in ℓ of the electron involved in the transition satisfies the following relation: $\Delta\ell = \pm 1$.

3. Changes in
   $s_{12}, \ell_{12}, j_{12}$, and $m_{j_{12}}$
   satisfy the following rules:
   $\Delta s_{12} = 0$
   $\Delta \ell_{12} = 0, \pm 1$
   $\Delta j_{12} = 0, \pm 1$ (but not $j_{12} = 0$ to $j_{12} = 0$)
   $\Delta m_{j_{12}} = 0, \pm 1$ (but not $m_{j_{12}} = 0$ to $m_{j_{12}} = 0$ when $\Delta j_{12} = 0$)

Zeeman effect:

$$\Delta E = -\overline{\mu} \bullet \overline{B} = \mu_b B g m_j$$

Landé *g* factor:

$$g = 1 + \frac{j(j+1) + s(s+1) - \ell(\ell+1)}{2j(j+1)}$$

Magnetic moment:

$$\bar{\mu} = \bar{\mu}_L + \bar{\mu}_S = -\frac{\mu_b}{\hbar}\left\{\bar{L}_{tot} + 2\bar{S}_{tot}\right\}$$

Characteristic temperature:

$$\theta = \frac{h\nu_{\max}}{k}$$

Boltzmann distribution:

$$n(\varepsilon) = Ae^{-\varepsilon/kT}$$

Bose distribution:

$$n(\varepsilon) = \frac{1}{e^{\alpha}e^{\varepsilon/kT} - 1}$$

Fermi distribution:

$$n(\varepsilon) = \frac{1}{e^{\alpha}e^{\varepsilon/kT} + 1} = \frac{1}{e^{(\varepsilon - \varepsilon_f)/kT} + 1}$$

Average energy of the bosons in a Bose condensate:

$$\bar{\varepsilon} \approx \frac{3}{2}kT\left\{1 - \frac{1}{2^{5/2}}e^{-\alpha}\right\}$$

Average energy of the fermions in a Fermi gas:

$$\bar{\varepsilon} \approx \frac{3}{2}kT\left\{1 + \frac{1}{2^{5/2}}e^{-\alpha}\right\}$$

Table 11-1 Comparison of the Three Distribution Functions

	Boltzmann	Bose	Fermi
Basic characteristic	Applies to distinguishable particles	Applies to indistinguishable particles not obeying the exclusion principle	Applies to indistinguishable particles obeying the exclusion principle
Example of system	Distinguishable particles, or approximation to quantum distributions at $\mathscr{E} \gg kT$	Bosons—identical particles of zero or integral spin	Fermions—identical particles of odd half integral spin
Eigenfunctions of particles	No symmetry requirements	Symmetric under exchange of particle labels	Antisymmetric under exchange of particle labels
Distribution function	$Ae^{-\mathscr{E}/kT}$	$\dfrac{1}{e^{\alpha}e^{\mathscr{E}/kT}-1}$	$\dfrac{1}{e^{(\mathscr{E}-\mathscr{E}_F)/kT}+1}$
Behavior of distribution function versus $\mathscr{E}/kT$	Exponential	For $\mathscr{E} \gg kT$, exponential For $\mathscr{E} \ll kT$, lies above Boltzmann	For $\mathscr{E} \gg kT$, exponential where $\mathscr{E} \gg \mathscr{E}_F$ If $\mathscr{E}_F \gg kT$, decreases abruptly near $\mathscr{E}_F$
Specific problems applied to in this chapter	Gases at essentially any temperature; modes of vibration in an isothermal enclosure	Photon gas (cavity radiation); phonon gas (heat capacity); liquid helium	Electron gas (electronic specific heat, contact potential, thermionic emission)

Electric quadrupole moment:

$$q=\int \rho\left(x,y,z\right)\left(3z^2-\left(x^2+y^2+z^2\right)\right)dz$$

Serber potential:

$$V_{Serber}=\frac{1}{2}V\left(r\right)\left(1+\left(-1\right)^{\ell}\right)$$

Isospin for nucleons:

Protons and neutrons are isospin ½ particle.

$T_z = \frac{1}{2}$ for protons

$T_z = -\frac{1}{2}$ for neutrons

Particle families:

Baryons: composite particles with three quarks.

Mesons: composite particles with a quark – anti-quark pair.

Leptons: elementary particles.

Baryon number is a conserved quantity.

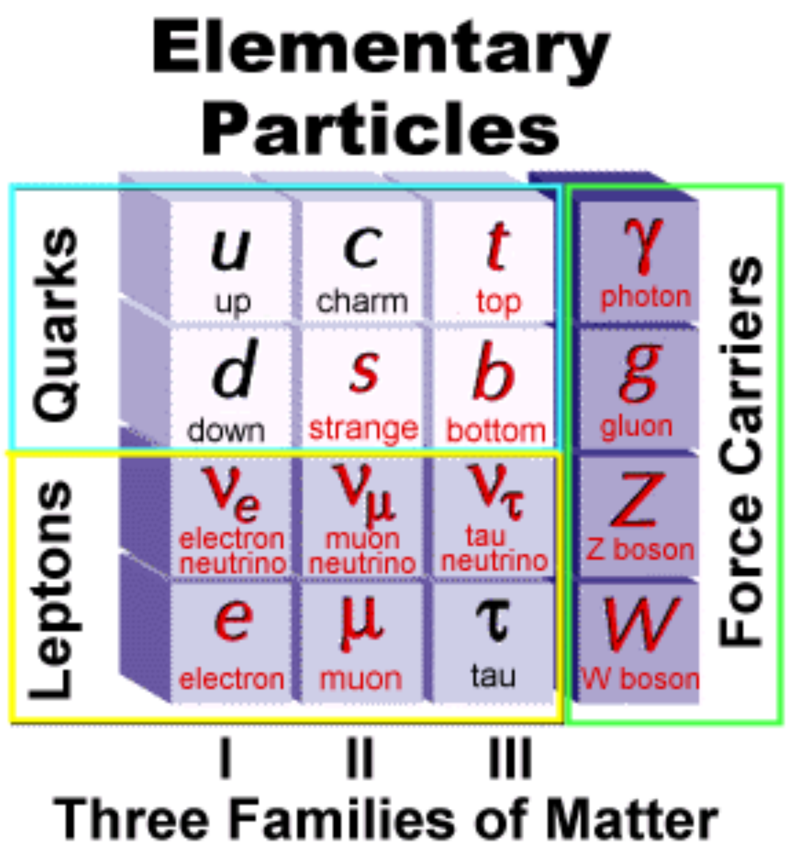
The electron lepton number, the muon lepton number, and the tau lepton number are conserved quantities

Electric charge:

$$Q = T_z + \frac{1}{2}\left(B + S + C + \beta + \tau\right)$$

Table 17-3. Applicability of the Conservation Laws to the Observed Interactions ("yes" Means Conserved; "no" Means Not Conserved)

Quantity Conserved	Strong	Electro-magnetic	Weak
Energy	yes	yes	yes
Linear momentum	yes	yes	yes
Angular momentum	yes	yes	yes
Charge	yes	yes	yes
Electronic lepton number	yes	yes	yes
Muonic lepton number	yes	yes	yes
Tauonic lepton number	yes	yes	yes
Baryon number	yes	yes	yes
Isospin magnitude	yes	no	no ($\Delta T = 1/2$ for nonleptonic)
Isospin z component	yes	yes	no ($\Delta T_z = 1/2$ for nonleptonic)
Strangeness	yes	yes	no ($\Delta S = 1$)
Parity	yes	yes	no
Charge conjugation	yes	yes	no
Time reversal (or CP)	yes	yes	yes (But 10^{-3} violation in K^0 decay)



Baryons qqq and Antibaryons $\bar{q}\bar{q}\bar{q}$

Baryons are fermionic hadrons.
There are about 120 types of baryons.

Symbol	Name	Quark content	Electric charge	Mass GeV/c^2	Spin
p	proton	uud	1	0.938	1/2
$\bar{p}$	anti-proton	$\bar{u}\bar{u}\bar{d}$	–1	0.938	1/2
n	neutron	udd	0	0.940	1/2
Λ	lambda	uds	0	1.116	1/2
Ω^-	omega	sss	–1	1.672	3/2

Mesons $q\bar{q}$

Mesons are bosonic hadrons.
There are about 140 types of mesons.

Symbol	Name	Quark content	Electric charge	Mass GeV/c^2	Spin
π^+	pion	$u\bar{d}$	+1	0.140	0
K^-	kaon	$s\bar{u}$	–1	0.494	0
ρ^+	rho	$u\bar{d}$	+1	0.770	1
B^0	B-zero	$d\bar{b}$	0	5.279	0
η_c	eta-c	$c\bar{c}$	0	2 .980	0

Important additional information

Best baseball team in the USA:

Yankees

Best soccer team in the world:

AJAX

Best airline in the world:

KLM (Koninklijke Luchtvaart Maatschappij = Royal Dutch Airlines)

If in doubt, the correct answer may be:

Yankees, AJAX, the Netherlands, or KLM.

APPENDIX D

*Useful Formulas**

D.1 Binomial Expansion

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + \binom{n}{r}x^r + \cdots, \quad |x| < 1 \tag{D.1}$$

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + (-1)^r\binom{n}{r}x^r + \cdots, \quad |x| < 1 \tag{D.2}$$

where the **binomial coefficient** is

$$\binom{n}{r} \equiv \frac{n!}{(n-r)!\,r!} \tag{D.3}$$

Some particularly useful cases of the above are

$$(1 \pm x)^{1/2} = 1 \pm \frac{1}{2}x - \frac{1}{8}x^2 \pm \frac{1}{16}x^3 - \cdots \tag{D.4}$$

$$(1 \pm x)^{1/3} = 1 \pm \frac{1}{3}x - \frac{1}{9}x^2 \pm \frac{5}{81}x^3 - \cdots \tag{D.5}$$

*An extensive list may be found, for example, in Dwight (Dw61).

$$(1 \pm x)^{-1/2} = 1 \mp \frac{1}{2}x + \frac{3}{8}x^2 \mp \frac{5}{16}x^3 + \cdots \tag{D.6}$$

$$(1 \pm x)^{-1/3} = 1 \mp \frac{1}{3}x + \frac{2}{9}x^2 \mp \frac{14}{81}x^3 + \cdots \tag{D.7}$$

$$(1 \pm x)^{-1} = 1 \mp x + x^2 \mp x^3 + \cdots \tag{D.8}$$

$$(1 \pm x)^{-2} = 1 \mp 2x + 3x^2 \mp 4x^3 + \cdots \tag{D.9}$$

$$(1 \pm x)^{-3} = 1 \mp 3x + 6x^2 \mp 10x^3 + \cdots \tag{D.10}$$

For convergence of *all* the above series, we must have $|x| < 1$.

D.2 Trigonometric Relations

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \tag{D.11}$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \tag{D.12}$$

$$\sin 2A = 2 \sin A \cos A = \frac{2\tan A}{1 + \tan^2 A} \tag{D.13}$$

$$\cos 2A = 2\cos^2 A - 1 \tag{D.14}$$

$$\sin^2 \frac{A}{2} = \frac{1}{2}(1 - \cos A) \tag{D.15}$$

$$\cos^2 \frac{A}{2} = \frac{1}{2}(1 + \cos A) \tag{D.16}$$

$$\sin^2 A = \frac{1}{2}(1 - \cos 2A) \tag{D.17}$$

$$\sin^3 A = \frac{1}{4}(3 \sin A - \sin 3A) \tag{D.18}$$

$$\sin^4 A = \frac{1}{8}(3 - 4\cos 2A + \cos 4A) \tag{D.19}$$

$$\cos^2 A = \frac{1}{2}(1 + \cos 2A) \tag{D.20}$$

$$\cos^3 A = \frac{1}{4}(3\cos A + \cos 3A) \tag{D.21}$$

$$\cos^4 A = \frac{1}{8}(3 + 4\cos 2A + \cos 4A) \tag{D.22}$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \quad \text{(D.23)}$$

$$\tan^2\frac{A}{2} = \frac{1-\cos A}{1+\cos A} \quad \text{(D.24)}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \quad \text{(D.25)}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \text{(D.26)}$$

$$e^{ix} = \cos x + i \sin x \quad \text{(D.27)}$$

D.3 Trigonometric Series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \text{(D.28)}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \text{(D.29)}$$

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots, \quad |x| < \pi/2 \quad \text{(D.30)}$$

$$\sin^{-1} x = x + \frac{x^3}{6} + \frac{3}{40}x^5 + \cdots, \quad \begin{cases} |x| < 1 \\ |\sin^{-1} x| < \pi/2 \end{cases} \quad \text{(D.31)}$$

$$\cos^{-1} x = \frac{\pi}{2} - x - \frac{x^3}{6} - \frac{3}{40}x^5 - \cdots, \quad \begin{cases} |x| < 1 \\ 0 < \cos^{-1} x < \pi \end{cases} \quad \text{(D.32)}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \quad |x| < 1 \quad \text{(D.33)}$$

D.4 Exponential and Logarithmic Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{(D.34)}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad |x| < 1, \quad x = 1 \quad \text{(D.35)}$$

$$\ln\left[\sqrt{(x^2/a^2) + 1} + (x/a)\right] = \sinh^{-1} x/a \quad \text{(D.36)}$$

$$= -\ln\left[\sqrt{(x^2/a^2) + 1} - (x/a)\right] \quad \text{(D.37)}$$

D.5 Complex Quantities

Cartesian form: $z = x + iy$, complex conjugate $z^* = x - iy$, $i = \sqrt{-1}$ (D.38)

Polar form:

$$z = |z| e^{i\theta} \quad \text{(D.39)}$$

$$z^* = |z| e^{-i\theta} \quad \text{(D.40)}$$

$$zz^* = |z|^2 = x^2 + y^2 \quad \text{(D.41)}$$

Real part of z: $\mathrm{Re}\, z = \frac{1}{2}(z + z^*) = x$ (D.42)

Imaginary part of z: $\mathrm{Im}\, z = -\frac{1}{2}(z - z^*) = y$ (D.43)

Euler's formula: $e^{i\theta} = \cos\theta + i \sin\theta$ (D.44)

D.6 Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{(D.45)}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{(D.46)}$$

$$\tanh x = \frac{e^{2x} - 1}{e^{2x} + 1} \quad \text{(D.47)}$$

$$\sin ix = i \sinh x \quad \text{(D.48)}$$

$$\cos ix = \cosh x \quad \text{(D.49)}$$

$$\sinh ix = i \sin x \quad \text{(D.50)}$$

$$\cosh ix = \cos x \quad \text{(D.51)}$$

$$\sinh^{-1} x = \tanh^{-1}\left(\frac{x}{\sqrt{x^2+1}}\right) \quad \text{(D.52)}$$

$$= \ln(x + \sqrt{x^2+1}) \quad \text{(D.53)}$$

$$= \cosh^{-1}(\sqrt{x^2+1}), \quad \begin{cases} > 0, & x > 0 \\ < 0, & x < 0 \end{cases} \quad \text{(D.54)}$$

$$\cosh^{-1} x = \pm \tanh^{-1}\left(\frac{\sqrt{x^2-1}}{x}\right), \quad x > 1 \quad \text{(D.55)}$$

$$= \pm \ln(x + \sqrt{x^2-1}), \quad x > 1 \quad \text{(D.56)}$$

$$\cosh^{-1} x = \pm \sinh^{-1}(\sqrt{x^2 - 1}), \quad x > 1 \tag{D.57}$$

$$\frac{d}{dy}\sinh y = \cosh y \tag{D.58}$$

$$\frac{d}{dy}\cosh y = \sinh y \tag{D.59}$$

$$\sinh(x_1 + x_2) = \sinh x_1 \cosh x_2 + \cosh x_1 \sinh x_2 \tag{D.60}$$

$$\cosh(x_1 + x_2) = \cosh x_1 \cosh x_2 + \sinh x_1 \sinh x_2 \tag{D.61}$$

$$\cosh^2 x - \sinh^2 x = 1 \tag{D.62}$$

PROBLEMS

D-1. Is it possible to ascribe a meaning to the inequality $z_1 < z_2$? Explain. Does the inequality $|z_1| < |z_2|$ have a different meaning?

D-2. Solve the following equations:
(a) $z^2 + 2z + 2 = 0$ **(b)** $2z^2 + z + 2 = 0$

D-3. Express the following in polar form:
(a) $z_1 = i$ **(b)** $z_2 = -1$
(c) $z_3 = 1 + i\sqrt{3}$ **(d)** $z_4 = 1 + 2i$
(e) Find the product z_1z_2 **(f)** Find the product z_1z_3
(g) Find the product z_3z_4

D-4. Express $(z^2 - 1)^{-1/2}$ in polar form.

D-5. If the function $w = \sin^{-1} z$ is defined as the inverse of $z = \sin w$, then use the Euler relation for $\sin w$ to find an equation for $\exp(iw)$. Solve this equation and obtain the result

$$w = \sin^{-1} z = -i \ln\left(iz + \sqrt{1 - z^2}\right)$$

D-6. Show that

$$y = Ae^{ix} + Be^{-ix}$$

can be written as

$$y = C\cos(x - \delta)$$

where A and B are *complex* but where C and δ are *real*.

D-7. Show that
(a) $\sinh(x_1 + x_2) = \sinh x_1 \cosh x_2 + \cosh x_1 \sinh x_2$
(b) $\cosh(x_1 + x_2) = \cosh x_1 \cosh x_2 + \sinh x_1 \sinh x_2$

APPENDIX E

*Useful Integrals**

E.1 Algebraic Functions

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right), \quad \left|\tan^{-1}\left(\frac{x}{a}\right)\right| < \frac{\pi}{2} \tag{E.1}$$

$$\int \frac{x\,dx}{a^2 + x^2} = \frac{1}{2}\ln(a^2 + x^2) \tag{E.2}$$

$$\int \frac{dx}{x(a^2 + x^2)} = \frac{1}{2a^2}\ln\left(\frac{x^2}{a^2 + x^2}\right) \tag{E.3}$$

$$\int \frac{dx}{a^2x^2 - b^2} = \frac{1}{2ab}\ln\left(\frac{ax - b}{ax + b}\right) \tag{E.4a}$$

$$= -\frac{1}{ab}\coth^{-1}\left(\frac{ax}{b}\right), \quad a^2x^2 > b^2 \tag{E.4b}$$

$$= -\frac{1}{ab}\tanh^{-1}\left(\frac{ax}{b}\right), \quad a^2x^2 < b^2 \tag{E.4c}$$

$$\int \frac{dx}{\sqrt{a + bx}} = \frac{2}{b}\sqrt{a + bx} \tag{E.5}$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln(x + \sqrt{x^2 + a^2}) \tag{E.6}$$

*This list is confined to those (nontrivial) integrals that arise in the text and in the problems. Extremely useful compilations are, for example, Pierce and Foster (Pi57) and Dwight (Dw61).

$$\int \frac{x^2 dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \tag{E.7}$$

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}} = \frac{1}{\sqrt{a}} \ln(2\sqrt{a}\sqrt{ax^2 + bx + c} + 2ax + b), \quad a > 0 \tag{E.8a}$$

$$= \frac{1}{\sqrt{a}}\sinh^{-1}\left(\frac{2ax + b}{\sqrt{4ac - b^2}}\right), \quad \begin{cases} a > 0 \\ 4ac > b^2 \end{cases} \tag{E.8b}$$

$$= -\frac{1}{\sqrt{-a}}\sin^{-1}\left(\frac{2ax + b}{\sqrt{b^2 - 4ac}}\right), \quad \begin{cases} a < 0 \\ b^2 > 4ac \\ |2ax + b| < \sqrt{b^2 - 4ac} \end{cases} \tag{E.8c}$$

$$\int \frac{x\,dx}{\sqrt{ax^2 + bx + c}} = \frac{1}{a}\sqrt{ax^2 + bx + c} - \frac{b}{2a}\int \frac{dx}{\sqrt{ax^2 + bx + c}} \tag{E.9}$$

$$\int \frac{dx}{x\sqrt{ax^2 + bx + c}} = -\frac{1}{\sqrt{c}}\sinh^{-1}\left(\frac{bx + 2c}{|x|\sqrt{4ac - b^2}}\right), \quad \begin{cases} c > 0 \\ 4ac > b^2 \end{cases} \tag{E.10a}$$

$$= \frac{1}{\sqrt{-c}}\sin^{-1}\left(\frac{bx + 2c}{|x|\sqrt{b^2 - 4ac}}\right), \quad \begin{cases} c < 0 \\ b^2 > 4ac \end{cases} \tag{E.10b}$$

$$= -\frac{1}{\sqrt{c}}\ln\left(\frac{2\sqrt{c}}{x}\sqrt{ax^2 + bx + c} + \frac{2c}{x} + b\right), \quad c > 0 \tag{E.10c}$$

$$\int \sqrt{ax^2 + bx + c}\,dx = \frac{2ax + b}{4a}\sqrt{ax^2 + bx + c} + \frac{4ac - b^2}{8a}\int \frac{dx}{\sqrt{ax^2 + bx + c}} \tag{E.11}$$

E.2 Trigonometric Functions

$$\int \sin^2 x\,dx = \frac{x}{2} - \frac{1}{4}\sin 2x \tag{E.12}$$

$$\int \cos^2 x\,dx = \frac{x}{2} + \frac{1}{4}\sin 2x \tag{E.13}$$

$$\int \frac{dx}{a + b\sin x} = \frac{2}{\sqrt{a^2 - b^2}}\tan^{-1}\left[\frac{a\tan(x/2) + b}{\sqrt{a^2 - b^2}}\right], \quad a^2 > b^2 \tag{E.14}$$

$$\int \frac{dx}{a + b\cos x} = \frac{2}{\sqrt{a^2 - b^2}}\tan^{-1}\left[\frac{(a - b)\tan(x/2)}{\sqrt{a^2 - b^2}}\right], \quad a^2 > b^2 \tag{E.15}$$

$$\int \frac{dx}{(a + b\cos x)^2} = \frac{b\sin x}{(b^2 - a^2)(a + b\cos x)} - \frac{a}{b^2 - a^2}\int \frac{dx}{a + b\cos x} \tag{E.16}$$

$$\int \tan x\,dx = -\ln|\cos x| \tag{E.17a}$$

$$\int \tanh x\,dx = \ln\cosh x \tag{E.17b}$$

$$\int e^{ax}\sin x\,dx = \frac{e^{ax}}{a^2 + 1}(a\sin x - \cos x) \tag{E.18a}$$

$$\int e^{ax}\sin^2 x\,dx = \frac{e^{ax}}{a^2 + 4}\left(a\sin^2 x - 2\sin x\cos x + \frac{2}{a}\right) \tag{E.18b}$$

$$\int_{-\infty}^{\infty} e^{-ax^2}dx = \sqrt{\pi/a} \tag{E.18c}$$

E.3 Gamma Functions

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx \tag{E.19a}$$

$$= \int_0^1 [\ln(1/x)]^{n-1}dx \tag{E.19b}$$

$$\Gamma(n) = (n - 1)!, \quad \text{for } n = \text{positive integer} \tag{E.19c}$$

$$n\Gamma(n) = \Gamma(n + 1) \tag{E.20}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{E.21}$$

$$\Gamma(1) = 1 \tag{E.22}$$

$$\Gamma\left(1\frac{1}{4}\right) = 0.906 \tag{E.23}$$

$$\Gamma\left(1\frac{3}{4}\right) = 0.919 \tag{E.24}$$

$$\Gamma(2) = 1 \tag{E.25}$$

$$\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\sqrt{\pi}}{n}\frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)} \tag{E.26}$$

$$\int_0^1 x^m(1-x^2)^n\,dx = \frac{\Gamma(n+1)\Gamma\left(\frac{m+1}{2}\right)}{2\Gamma\left(n+\frac{m+3}{2}\right)} \tag{E.27a}$$

$$\int_0^{\pi/2} \cos^n x\,dx = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}, \quad n > -1 \tag{E.27b}$$

APPENDIX F

Differential Relations in Different Coordinate Systems

F.1 Rectangular Coordinates

$$\mathbf{grad}\,U = \nabla U = \sum_i \mathbf{e}_i \frac{\partial U}{\partial x_i} \tag{F.1}$$

$$\text{div}\,\mathbf{A} = \nabla\cdot\mathbf{A} = \sum_i \frac{\partial A_i}{\partial x_i} \tag{F.2}$$

$$\text{curl}\,\mathbf{A} = \nabla\times\mathbf{A} = \sum_{i,j,k} \varepsilon_{ijk}\frac{\partial A_k}{\partial x_j}\mathbf{e}_i \tag{F.3}$$

$$\nabla^2 U = \nabla\cdot\nabla U = \sum_i \frac{\partial^2 U}{\partial x_i^2} \tag{F.4}$$

F.2 Cylindrical Coordinates

Refer to Figures F-1 and F-2.

$$x_1 = r\cos\phi, \quad x_2 = r\sin\phi, \quad x_3 = z \tag{F.5}$$

$$r = \sqrt{x_1^2 + x_2^2}, \quad \phi = \tan^{-1}\frac{x_2}{x_1}, \quad z = x_3 \tag{F.6}$$

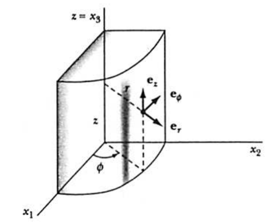


FIGURE F-1

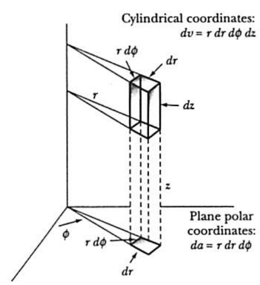


FIGURE F-2

$$ds^2 = dr^2 + r^2\,d\phi^2 + dz^2 \tag{F.7}$$

$$dv = r\,dr\,d\phi\,dz \tag{F.8}$$

$$\textbf{grad}\ \psi = \nabla\psi = \mathbf{e}_r \frac{\partial\psi}{\partial r} + \mathbf{e}_\phi \frac{1}{r}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_z \frac{\partial\psi}{\partial z} \tag{F.9}$$

$$\text{div}\ \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z} \tag{F.10}$$

$$\textbf{curl}\ \mathbf{A} = \mathbf{e}_r\left(\frac{1}{r}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right) + \mathbf{e}_\phi\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \mathbf{e}_z\left(\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi) - \frac{1}{r}\frac{\partial A_r}{\partial\phi}\right) \tag{F.11}$$

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} \tag{F.12}$$

F.3 Spherical Coordinates

Refer to Figures F-3 and F-4

$$x_1 = r\sin\theta\cos\phi, \quad x_2 = r\sin\theta\sin\phi, \quad x_3 = r\cos\theta \tag{F.13}$$

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad \theta = \cos^{-1}\frac{x_3}{r}, \quad \phi = \tan^{-1}\frac{x_2}{x_1} \tag{F.14}$$

$$ds^2 = dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2 \tag{F.15}$$

$$dv = r^2\sin\theta\,dr\,d\theta\,d\phi \tag{F.16}$$

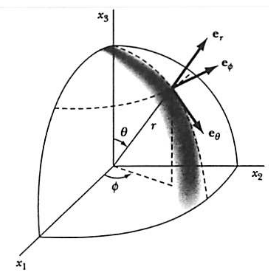


FIGURE F-3

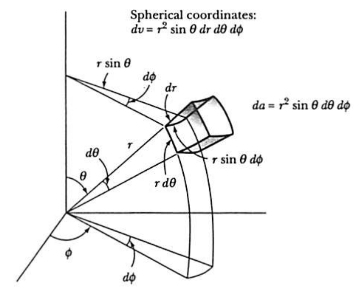


FIGURE F-4

$$\mathbf{grad}\,\psi = \nabla\psi = \mathbf{e}_r\frac{\partial\psi}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_\phi\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi} \tag{F.17}$$

$$\mathrm{div}\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta} + \frac{\partial A_\phi}{\partial\phi} \tag{F.18}$$

$$\mathbf{curl\ A} = \mathbf{e}_r\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right]$$

$$+ \mathbf{e}_\theta\frac{1}{r\sin\theta}\left[\frac{\partial A_r}{\partial\phi} - \sin\theta\frac{\partial}{\partial r}(rA_\phi)\right] + \mathbf{e}_\phi\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right] \tag{F.19}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} \tag{F.20}$$

APPENDIX

A Mathematical Formulas

A–1 Quadratic Formula

If $ax^2 + bx + c = 0$

then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

A–2 Binomial Expansion

$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)}{2!}x^2 \pm \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

$$(x + y)^n = x^n\left(1 + \frac{y}{x}\right)^n = x^n\left(1 + n\frac{y}{x} + \frac{n(n-1)}{2!}\frac{y^2}{x^2} + \cdots\right)$$

A–3 Other Expansions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$$

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2}{15}\theta^5 + \cdots \qquad |\theta| < \frac{\pi}{2}$$

In general: $f(x) = f(0) + \left(\dfrac{df}{dx}\right)_0 x + \left(\dfrac{d^2f}{dx^2}\right)_0 \dfrac{x^2}{2!} + \cdots$

A–4 Exponents

$$(a^n)(a^m) = a^{n+m} \qquad \frac{1}{a^n} = a^{-n}$$

$$(a^n)(b^n) = (ab)^n \qquad a^n a^{-n} = a^0 = 1$$

$$(a^n)^m = a^{nm} \qquad a^{\frac{1}{2}} = \sqrt{a}$$

A–5 Areas and Volumes

Object	Surface area	Volume
Circle, radius r	πr^2	—
Sphere, radius r	$4\pi r^2$	$\frac{4}{3}\pi r^3$
Right circular cylinder, radius r, height h	$2\pi r^2 + 2\pi rh$	$\pi r^2 h$
Right circular cone, radius r, height h	$\pi r^2 + \pi r\sqrt{r^2 + h^2}$	$\frac{1}{3}\pi r^2 h$

A-1

A–8 Vectors

Vector addition is covered in Sections 3–2 to 3–5.
Vector multiplication is covered in Sections 3–3, 7–2, and 11–2.

A–9 Trigonometric Functions and Identities

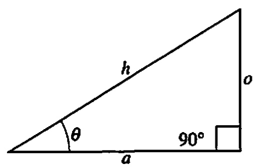


FIGURE A–5

The trigonometric functions are defined as follows (see Fig. A–5, o = side opposite, a = side adjacent, h = hypotenuse. Values are given in Table A–2):

$$\sin\theta = \frac{o}{h} \qquad \csc\theta = \frac{1}{\sin\theta} = \frac{h}{o}$$

$$\cos\theta = \frac{a}{h} \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{h}{a}$$

$$\tan\theta = \frac{o}{a} = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{a}{o}$$

and recall that

$$a^2 + o^2 = h^2 \qquad \text{[Pythagorean theorem].}$$

FIGURE A–6

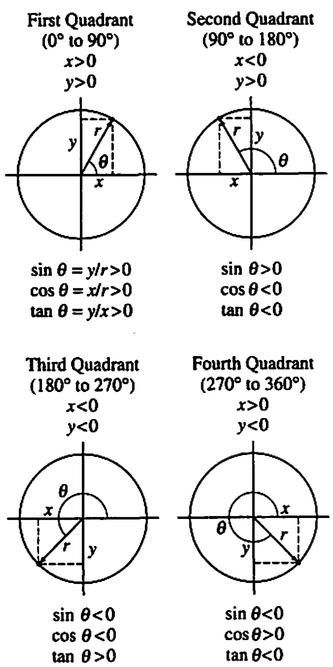


Figure A–6 shows the signs (+ or −) that cosine, sine, and tangent take on for angles θ in the four quadrants (0° to 360°). Note that angles are measured counterclockwise from the x axis as shown; negative angles are measured from *below* the x axis, clockwise: for example, $-30° = +330°$, and so on.

The following are some useful identities among the trigonometric functions:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sec^2\theta - \tan^2\theta = 1, \quad \csc^2\theta - \cot^2\theta = 1$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$

$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$

$$\sin(180° - \theta) = \sin\theta$$

$$\cos(180° - \theta) = -\cos\theta$$

$$\sin(90° - \theta) = \cos\theta$$

$$\cos(90° - \theta) = \sin\theta$$

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = -\tan\theta$$

$$\sin\tfrac{1}{2}\theta = \sqrt{\frac{1-\cos\theta}{2}}, \quad \cos\tfrac{1}{2}\theta = \sqrt{\frac{1+\cos\theta}{2}}, \quad \tan\tfrac{1}{2}\theta = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

$$\sin A \pm \sin B = 2\sin\left(\frac{A \pm B}{2}\right)\cos\left(\frac{A \mp B}{2}\right).$$

FIGURE A–7

For any triangle (see Fig. A–7):

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c} \qquad \text{[Law of sines]}$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma. \qquad \text{[Law of cosines]}$$

Values of sine, cosine, tangent are given in Table A–2.

A-4 APPENDIX A

APPENDIX

B Derivatives and Integrals

Derivatives: General Rules

(See also Section 2–3.)

$$\frac{dx}{dx} = 1$$

$$\frac{d}{dx}[af(x)] = a\frac{df}{dx} \qquad [a = \text{constant}]$$

$$\frac{d}{dx}[f(x) + g(x)] = \frac{df}{dx} + \frac{dg}{dx}$$

$$\frac{d}{dx}[f(x)g(x)] = \frac{df}{dx}g + f\frac{dg}{dx}$$

$$\frac{d}{dx}[f(y)] = \frac{df}{dy}\frac{dy}{dx} \qquad [\text{chain rule}]$$

$$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)} \qquad \text{if } \frac{dy}{dx} \neq 0.$$

Derivatives: Particular Functions

$$\frac{da}{dx} = 0 \qquad [a = \text{constant}]$$

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}\sin ax = a\cos ax$$

$$\frac{d}{dx}\cos ax = -a\sin ax$$

$$\frac{d}{dx}\tan ax = a\sec^2 ax$$

$$\frac{d}{dx}\ln ax = \frac{1}{x}$$

$$\frac{d}{dx}e^{ax} = ae^{ax}$$

Indefinite Integrals: General Rules

(See also Section 7–3.)

$$\int dx = x$$

$$\int af(x)\,dx = a\int f(x)\,dx \qquad [a = \text{constant}]$$

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int u\,dv = uv - \int v\,du \qquad [\text{integration by parts: see also B}$$

A-6

Indefinite Integrals: Particular Functions

(An arbitrary constant can be added to the right side of each equation.)

$$\int a\,dx = ax \qquad [a = \text{constant}]$$

$$\int x^m\,dx = \frac{1}{m+1}x^{m+1} \qquad [m \neq -1]$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax$$

$$\int \tan ax\,dx = \frac{1}{a}\ln|\sec ax|$$

$$\int \frac{1}{x}\,dx = \ln x$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left(x + \sqrt{x^2 \pm a^2}\right)$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) = -\cos^{-1}\left(\frac{x}{a}\right) \qquad [\text{if } x^2 \leq a^2]$$

$$\int \frac{dx}{(x^2 \pm a^2)^{\frac{3}{2}}} = \frac{\pm x}{a^2\sqrt{x^2 \pm a^2}}$$

$$\int \frac{x\,dx}{(x^2 \pm a^2)^{\frac{3}{2}}} = \frac{-1}{\sqrt{x^2 \pm a^2}}$$

$$\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$$

$$\int xe^{-ax}\,dx = -\frac{e^{-ax}}{a^2}(ax + 1)$$

$$\int x^2e^{-ax}\,dx = -\frac{e^{-ax}}{a^3}(a^2x^2 + 2ax + 2)$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left(\frac{x - a}{x + a}\right) \qquad [x^2 > a^2]$$

$$= -\frac{1}{2a}\ln\left(\frac{a + x}{a - x}\right) \qquad [x^2 < a^2]$$

A Few Definite Integrals

$$\int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}$$

$$\int_0^\infty e^{-ax^2}\,dx = \sqrt{\frac{\pi}{4a}}$$

$$\int_0^\infty xe^{-ax^2}\,dx = \frac{1}{2a}$$

$$\int_0^\infty x^2e^{-ax^2}\,dx = \sqrt{\frac{\pi}{16a^3}}$$

$$\int_0^\infty x^3e^{-ax^2}\,dx = \frac{1}{2a^2}$$

$$\int_0^\infty x^{2n}e^{-ax^2}\,dx = \frac{1\cdot3\cdot5\cdots(2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}$$

Integration by Parts

Sometimes a difficult integral can be simplified by carefully choosing the functions u and v in the identity:

$$\int u\,dv = uv - \int v\,du. \qquad \text{[Integration by parts]}$$

This identity follows from the property of derivatives

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

or as differentials: $d(uv) = u\,dv + v\,du$.

For example $\int xe^{-x}\,dx$ can be integrated by choosing $u = x$ and $dv = e^{-x}dx$ in the "integration by parts" equation above:

$$\int xe^{-x}\,dx = (x)(-e^{-x}) + \int e^{-x}\,dx$$

$$= -xe^{-x} - e^{-x} = -(x + 1)e^{-x}.$$


Good Luck !