Classical Mechanics
Phy 235, Review, Exam 3.

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Exam # 3

- Exam # 3 will take place on Tuesday December 7 at 8.00 am – 9.30 am in B&L 109.
- The exam will cover the material in Chapters 8 – 10.
- The exam will have 4 questions:
  - Three questions will be analytical questions.
  - One question will be a conceptual question (including concepts related to the Yankees or the Netherlands).
- You will be provided with an equation sheet.

Time management

- Work no more than 10 – 15 minutes on each problem.
- Even if not finished, move on to the next problem.
- This will leave 15 minutes at the end to finish your problems and/or make correction.
- We can only give credit for what you write (not what you think).
- We can only give credit for what we can read (write neatly).

Warning.

- I cannot cover everything I discussed in lectures 12 – 17 in this review.
- If I skip over certain topics, it does not mean you should not understand that material.
- Your TAs will not see the exam until you see it.
- **NOTE**: answer the correct question in the correct booklet.

Overview

- Chapter 8: Central-Force Motion.
  - Sections 8.9 and 8.10 are not included.
- Chapter 9: Dynamics of System of Particles.
- Chapter 10: Motion in Noninertial Reference Frames.

Chapter 8.

Chapter 8.
Central-Force Motion.

- Many important problems in physics involve the motion of two bodies with a central force acting between them.
- Assume the potential depends on the position between the two objects.
- The Lagrangian can be written in terms of the coordinates of the two masses:

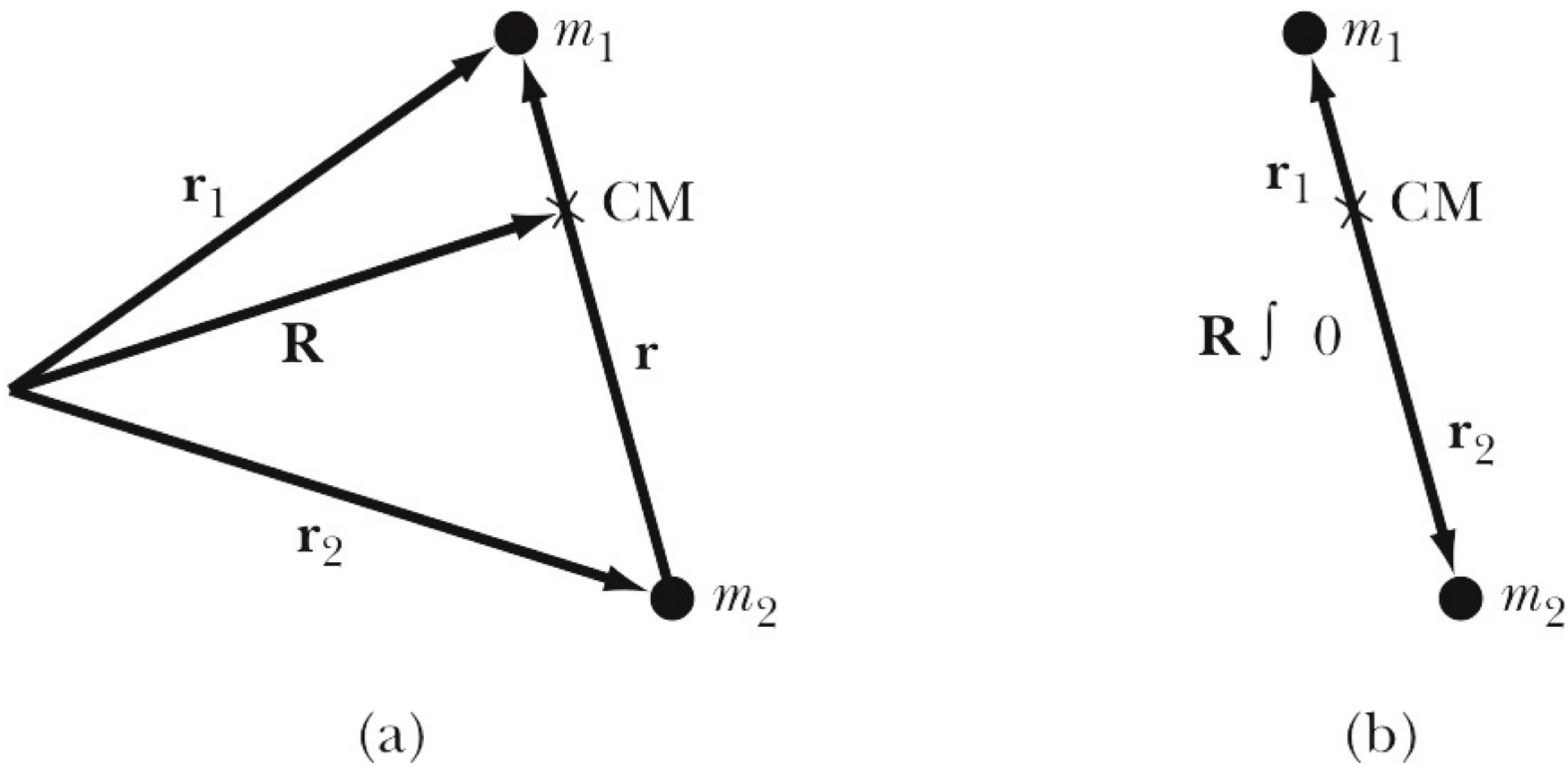
$$L = \frac{1}{2} m_1 \left|\dot{\bar{r}}_1\right|^2 + \frac{1}{2} m_2 \left|\dot{\bar{r}}_2\right|^2 - U\left(\bar{r}_1 - \bar{r}_2\right)$$

- Or in terms of their relative position:

$$L = \frac{1}{2} \mu \left|\dot{\bar{r}}\right|^2 - U\left(\bar{r}\right)$$

- Note: 2-body problem has been reduced to a one-body problem.

Changing a 2-body problem into a 1-body problem.



Conservation of angular momentum. Spherical symmetry: *U* only depends on *r*.

Starting from the Lagrangian:

$$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r)$$

we define the generalized momenta:

$$p_r = \frac{\partial L}{\partial \dot{r}} = \mu\dot{r}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2\dot{\theta}$$

The time derivatives of the generalized momenta are:

$$\dot{p}_r = \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r} = \mu r\dot{\theta}^2 - \frac{\partial U}{\partial r}$$

$$\dot{p}_\theta = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} = 0$$

Two-Body Central-Force Motion.

- For two-body central force problems, we showed:
  - Angular momentum was a conserved quantity.
  - Kepler's second law is a direct consequence of conservation of angular momentum.
  - Since the Lagrangian does not depend explicitly on time, energy is conserved.

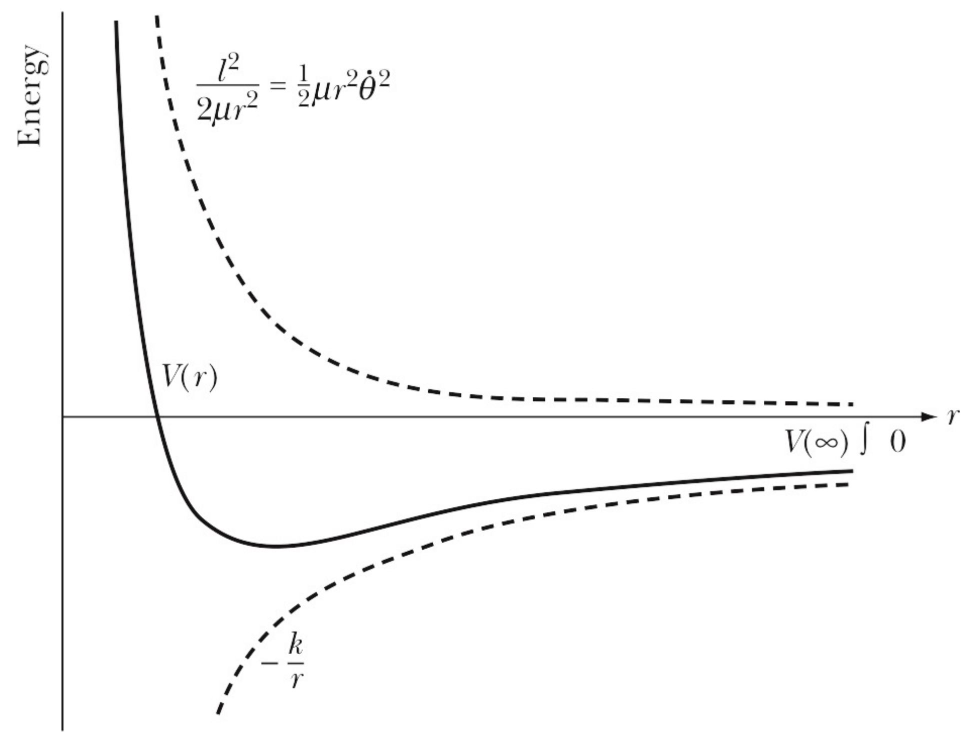
$$E = T + U = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + U(r) = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\left(\frac{l}{\mu r^2}\right)^2\right) + U(r) =$$

$$= \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{l^2}{\mu r^2} + U(r)$$

Modification to the potential energy.

The "effective" potential.

- The effective potential is composed of the real potential and the centrifugal potential energy.



- Observations:
  - The effective potential may show a dip that indicates that for certain energies, the orbit is bound.
  - For small distances, the effective force becomes repulsive.

Orbital motion: orbital properties.

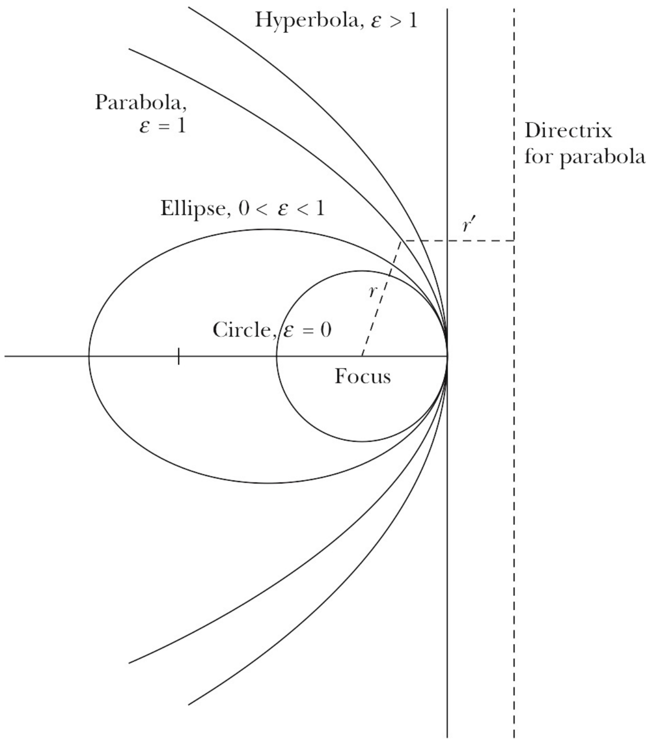
- Properties of the orbits can be found by solving the following integrals:

$$t = \int dt = \pm \int \frac{1}{\sqrt{\frac{2}{\mu}\left(E - U(r)\right) - \frac{l^2}{\mu^2 r^2}}} dr$$

$$\theta(r) = \int \frac{\dot{\theta}}{\dot{r}} dr = \pm \int \frac{l}{r^2 \sqrt{2\mu\left(E - U - \frac{l^2}{2\mu r^2}\right)}} dr$$

Orbital motion: orbital properties (shape).

- Properties:
  - $E > 0$: Hyperbola
  - $E = 0$: Parabola
  - $V_{min} < E < 0$: Ellipse
  - $E = V_{min}$: Circle



Problem 8.10

- Assume Earth's orbit to be circular and that the Sun's mass suddenly decreases by half? What orbit does the Earth then have?

Chapter 9.

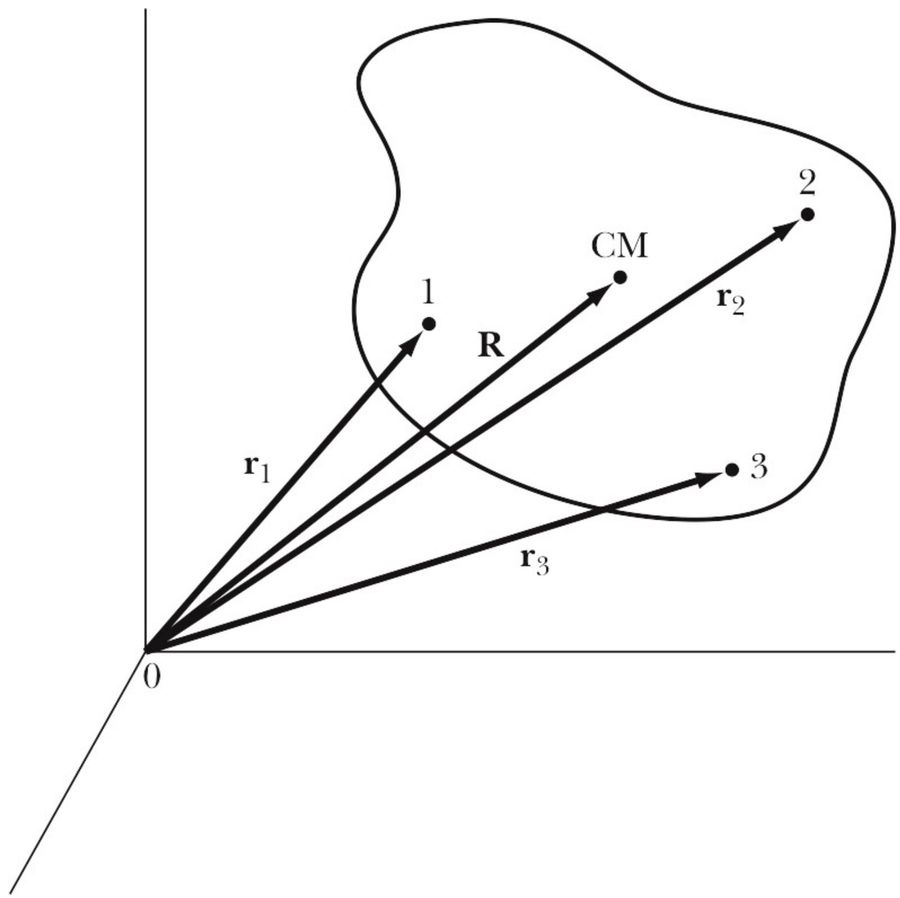
Center of Mass

- Definitions of center of mass:
  - Discrete mass distribution:

$$\bar{R}_{cm} = \frac{\sum_i m_i \bar{r}_i}{\sum_i m_i} = \frac{1}{M} \sum_i m_i \bar{r}_i$$

  - Continuous mass distribution:

$$\bar{R}_{cm} = \frac{1}{M} \int \bar{r}\, dm$$

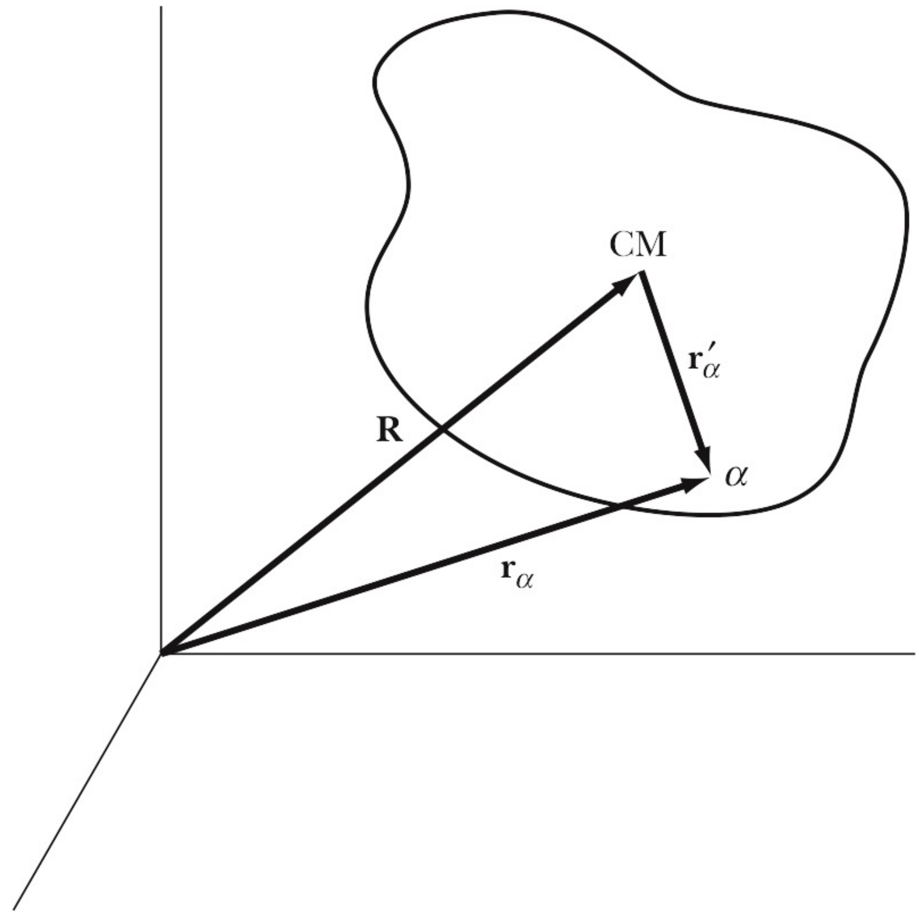


Linear Momentum.

- Linear momentum:

$$\bar{P} = \sum_i m_i \dot{\bar{r}}_i = \frac{d}{dt}\sum_i m_i \bar{r}_i =$$

$$= \frac{d}{dt}\left(M\bar{R}\right) = M\dot{\bar{R}}$$

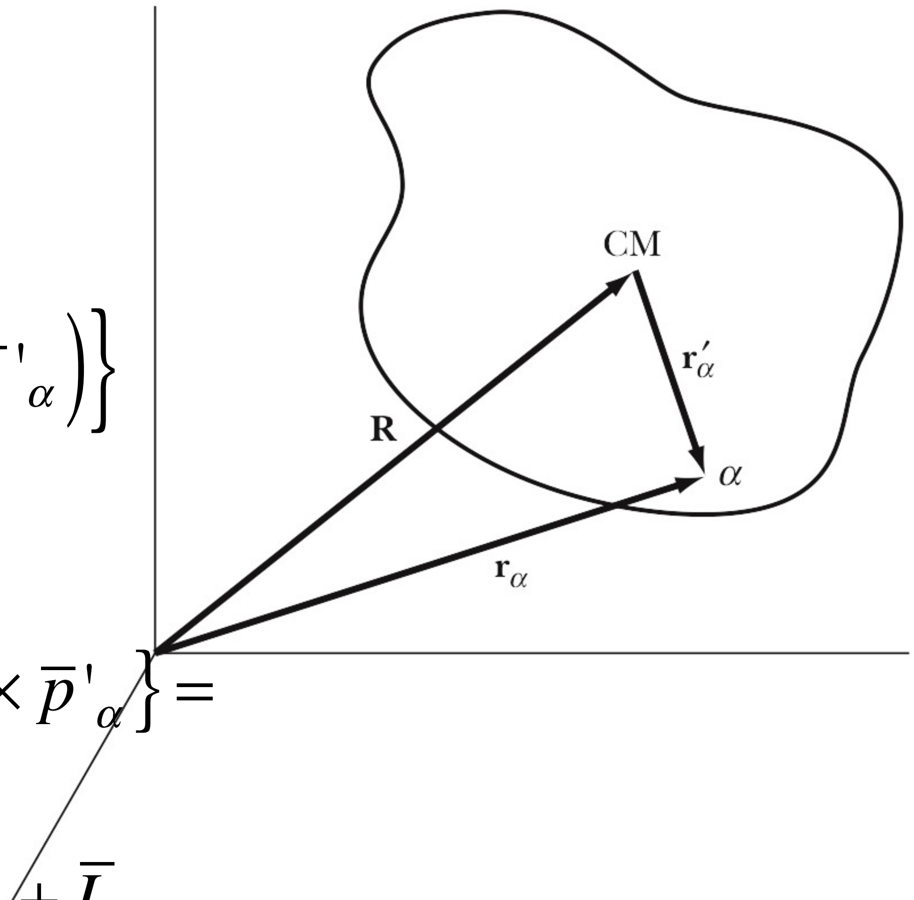


Angular Momentum.

- Angular momentum:

$$\bar{L} = \sum_\alpha \bar{L}_\alpha = \sum_\alpha \left\{ \bar{r}_\alpha \times m_\alpha \dot{\bar{r}}_\alpha \right\} =$$

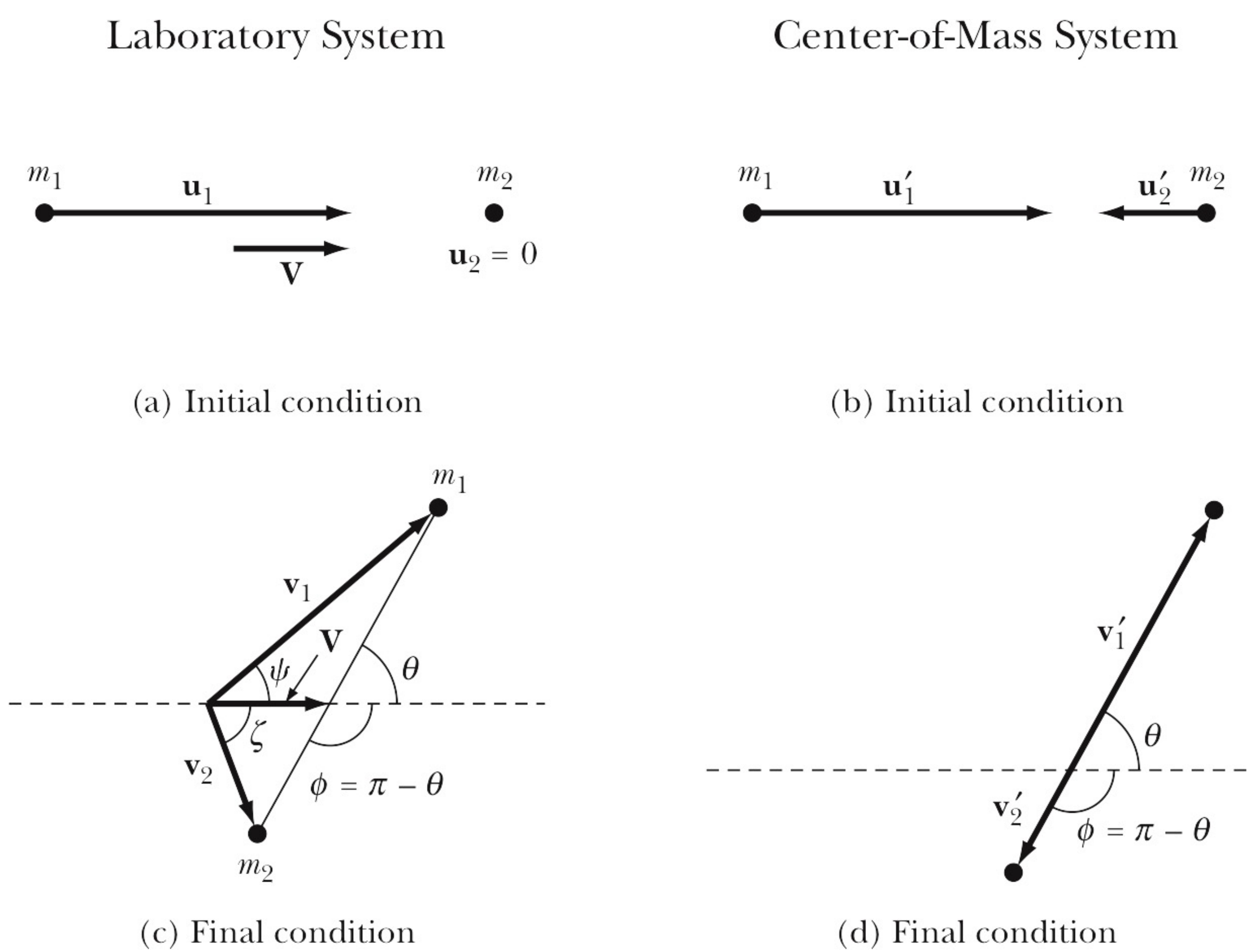
$$= \sum_\alpha \left\{ \left(\bar{R} + \bar{r}'_\alpha \right) \times m_\alpha \left(\dot{\bar{R}} + \dot{\bar{r}}'_\alpha \right) \right\}$$



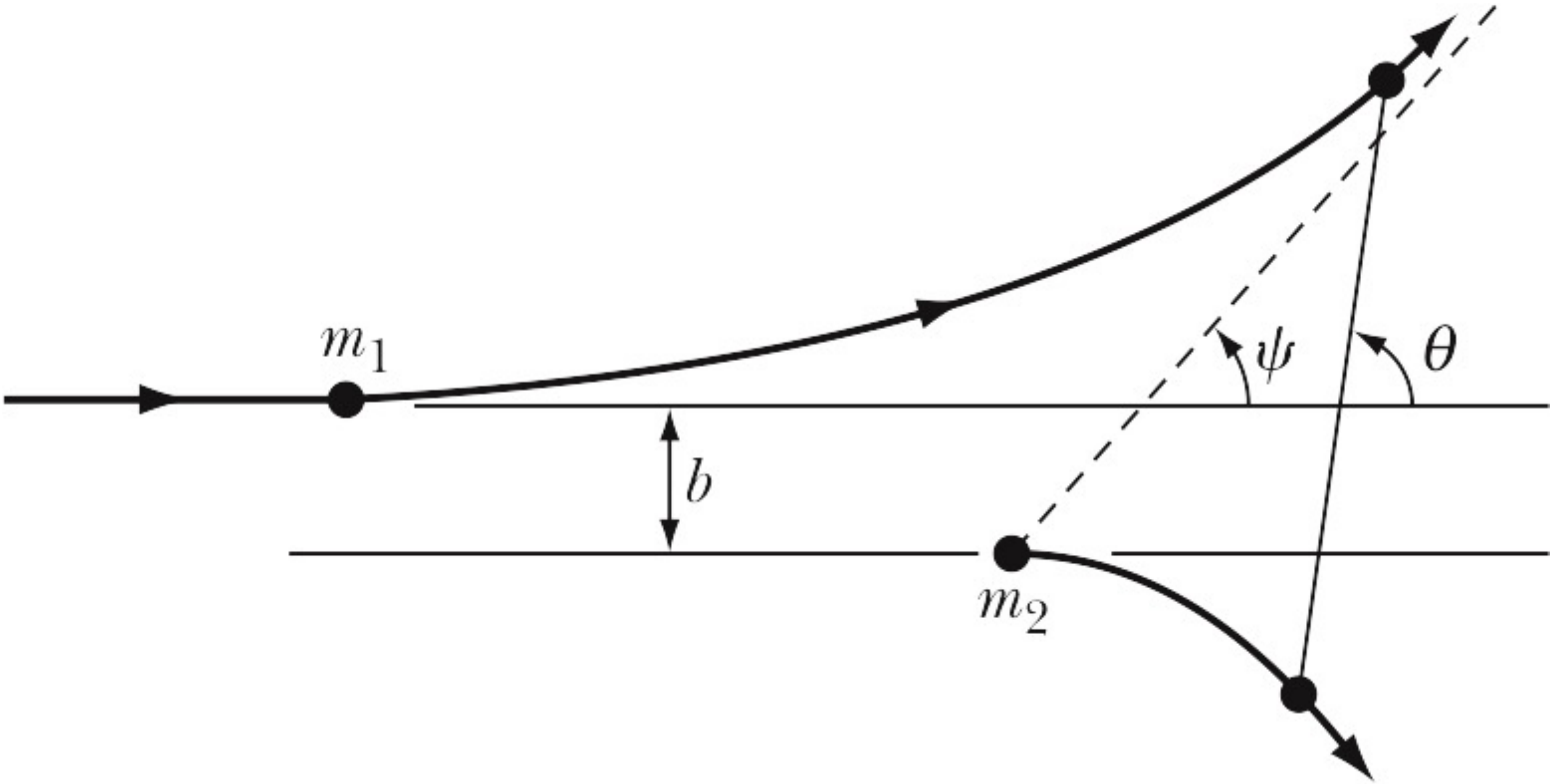
$$\bar{L} = \left(\bar{R} \times \dot{\bar{R}} \right) \sum_\alpha m_\alpha + \sum_\alpha \left\{ \bar{r}'_\alpha \times \bar{p}'_\alpha \right\} =$$

$$= \bar{R} \times \bar{P} + \sum_\alpha \bar{L}_{\alpha, wrt, cm} = \bar{L}_{cm} + \bar{L}_{wrt, cm}$$

Collisions.
Laboratory and Center-of-Mass Frames.



Impact parameter and scattering angle.



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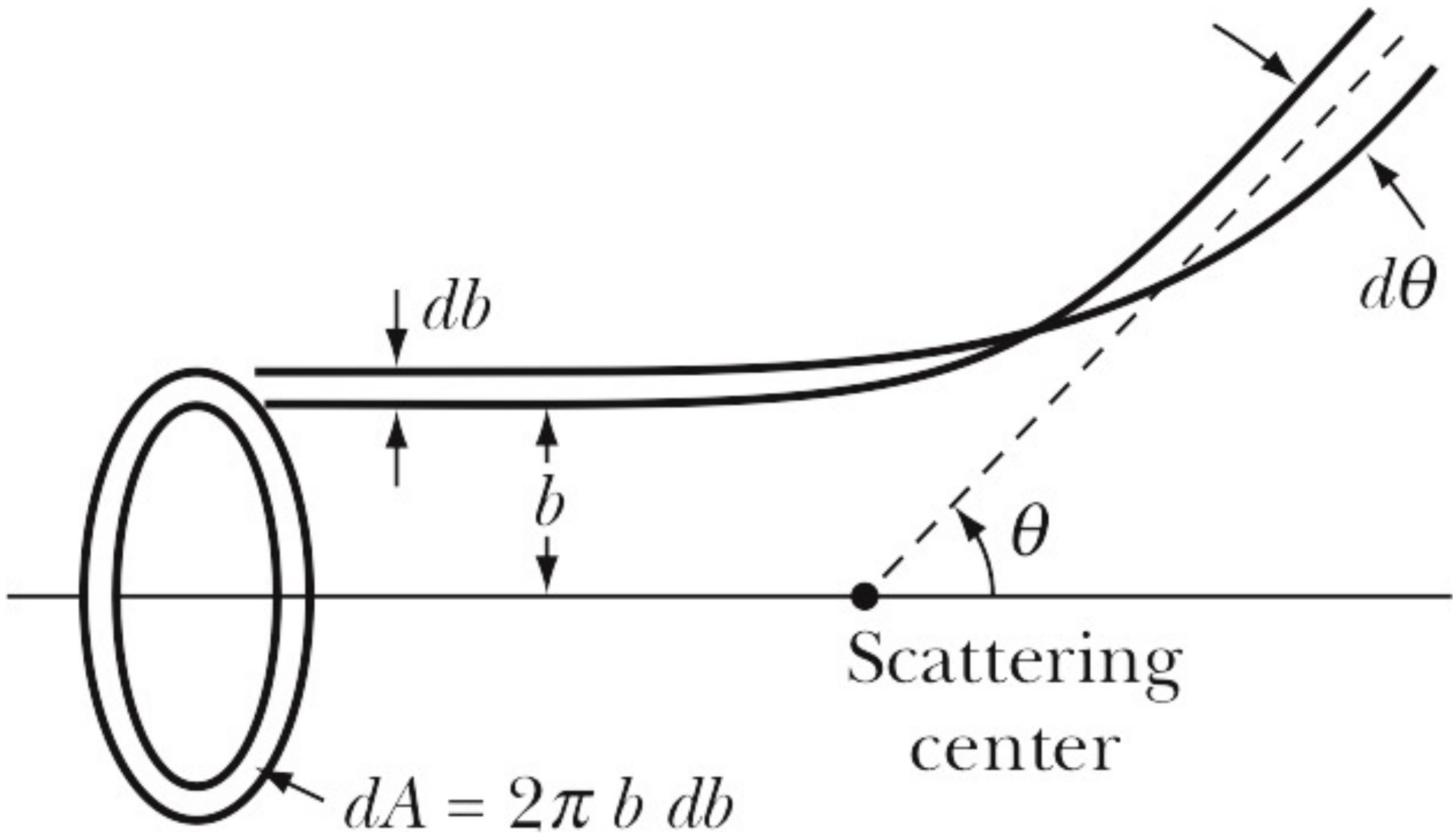
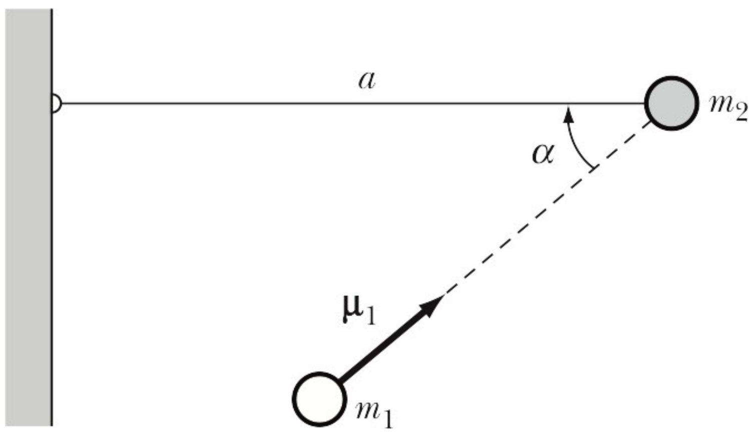


Problem 9.40.

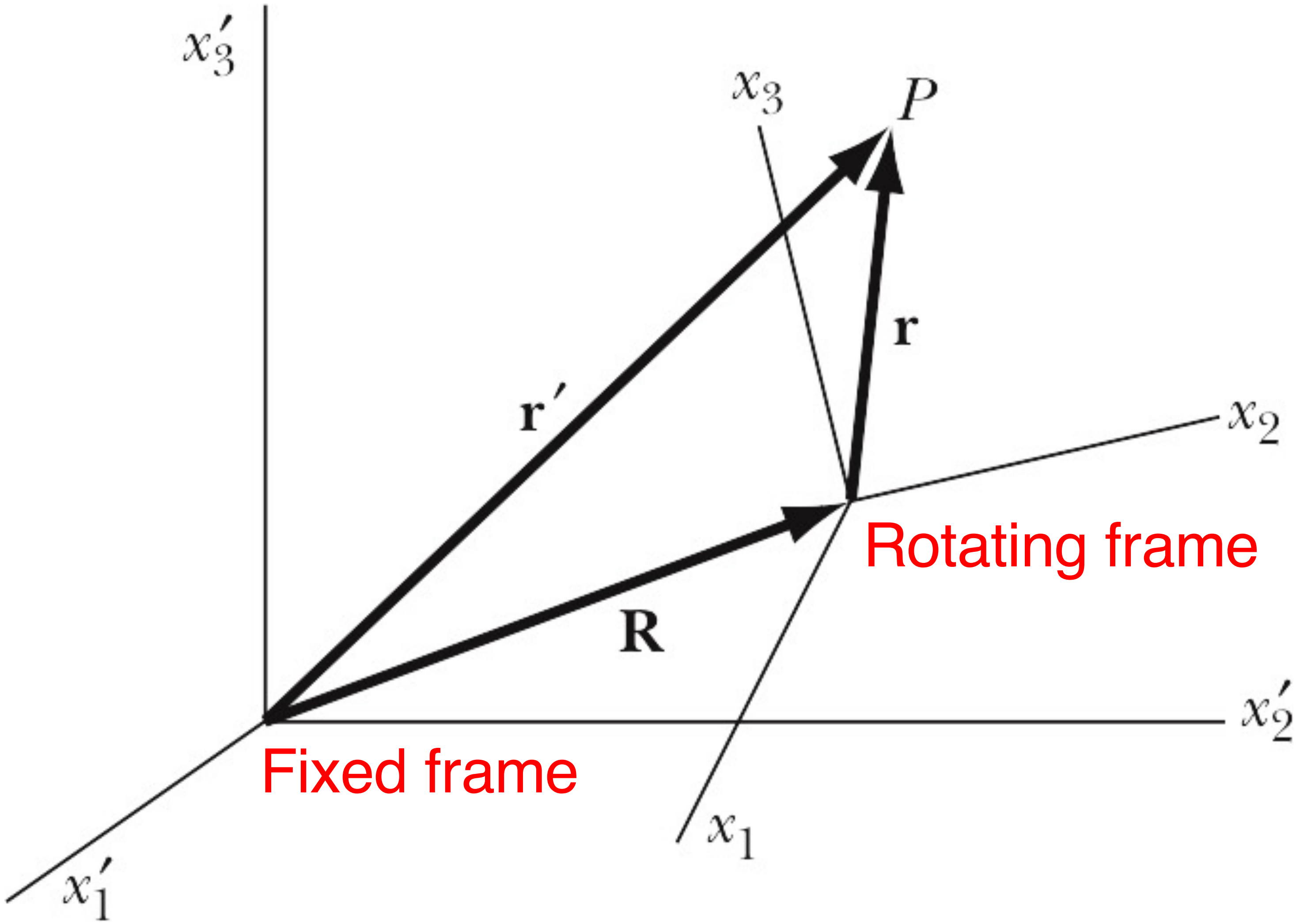
A particle of mass m_1 and velocity u_1, strikes head-on a particle of mass m_2 at rest. The coefficient of restitution is ε.



Particle m_2 is tied to a point a distance a away, as shown in the Figure. Find the velocity (magnitude and direction) of m_1 and m_2 after the collision.

Chapter 10.

Rotating Coordinate system.



The angular acceleration is the same in both reference frames.

- Relation between position vectors:

$$\left(\frac{d\bar{r}}{dt}\right)_{fixed} = \left(\frac{d\bar{r}}{dt}\right)_{rotating} + \bar{\omega}\times\bar{r}$$

- Relation between angular velocity vectors:

$$\left(\frac{d\bar{\omega}}{dt}\right)_{fixed} = \left(\frac{d\bar{\omega}}{dt}\right)_{rotating} + \bar{\omega}\times\bar{\omega} = \left(\frac{d\bar{\omega}}{dt}\right)_{rotating}$$

- Conclusion: the angular acceleration is the same in both reference frames.

Velocity in fixed (inertial) frame.

Velocity of the origin of the rotating frame.

$$v_f = \left(\frac{d\bar{r}\,'}{dt}\right)_{fixed} = \left(\frac{d\bar{R}}{dt}\right)_{fixed} + \left(\frac{d\bar{r}}{dt}\right)_{rotating} + \bar{\omega}\times\bar{r} = V + v_r + \bar{\omega}\times\bar{r}$$

Velocity in fixed frame.

Velocity in rotating frame.

Newton's laws in rotating reference frames.

- Only in the fixed reference frame can we use Newton's second law:

$$\bar{a}_f = \left(\frac{d\bar{v}_f}{dt} \right)_{fixed} = \frac{\bar{F}}{m}$$

- The acceleration in the fixed reference frame can also be expressed as:

$$\bar{a}_f = \left(\frac{d\bar{V}}{dt} \right)_{fixed} + \underbrace{\left(\frac{d\bar{v}_r}{dt} \right)_{rotating}}_{\text{Acceleration observed in rotating frame.}} + 2\bar{\omega} \times \bar{v}_r + \dot{\bar{\omega}} \times \bar{r} + \bar{\omega} \times \{\bar{\omega} \times \bar{r}\}$$

The effective force.

Using the **effective force**

Coriolis force.

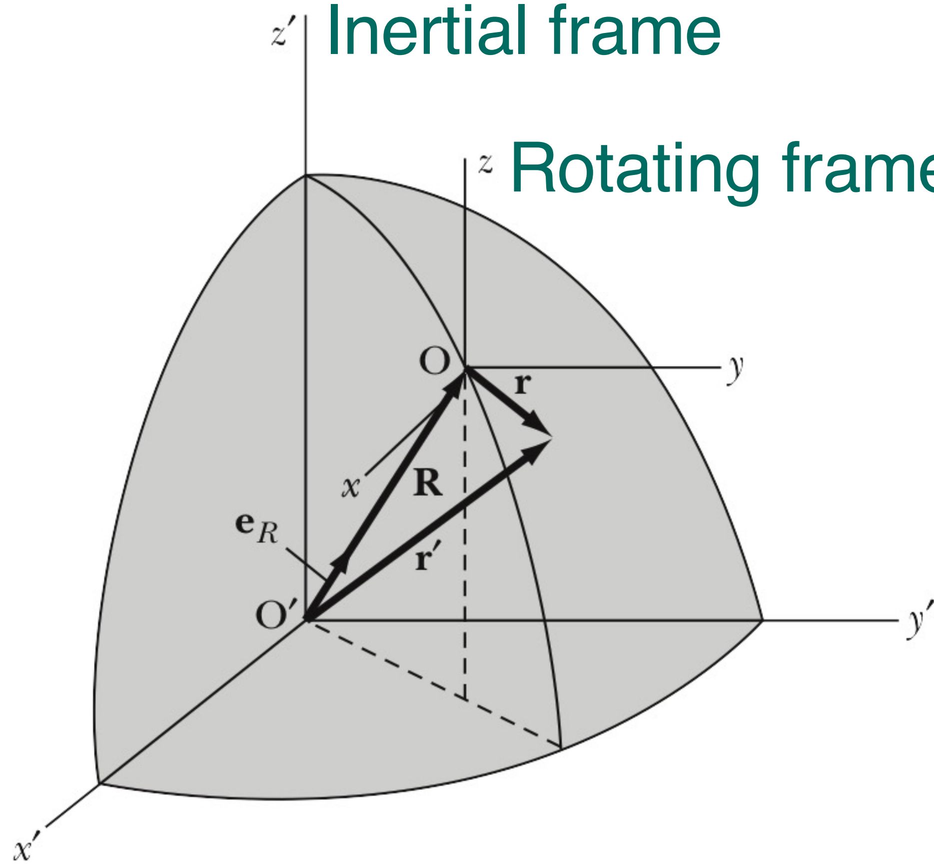
$$\bar{F}_{eff} = m\bar{a}_f - m\dot{\bar{\omega}} \times \bar{r} - 2m\bar{\omega} \times \bar{v}_r - m\bar{\omega} \times \{\bar{\omega} \times \bar{r}\}$$

Centripetal force.

an observer in the rotating frame will be able to determine the acceleration in the rotating frame by dividing the effective force by the mass of the object.

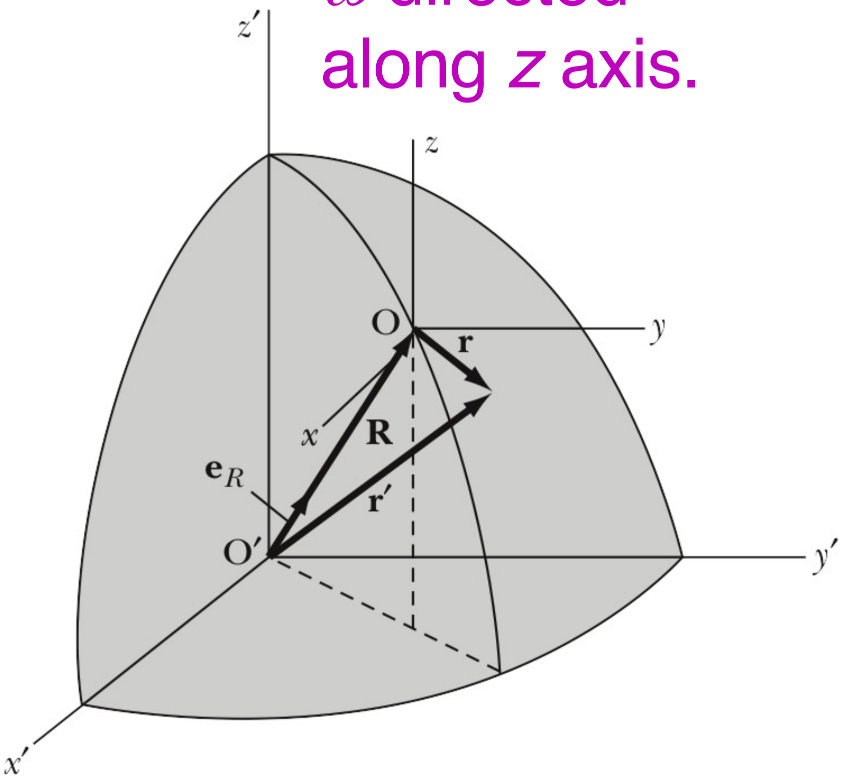
The centripetal force.

- The Earth is not a good inertial reference frame.
- The biggest "non-inertial" effect is due to the daily rotation around its axis.
- We use a rotating *xyz* frame, fixed on the surface of the Earth, and a fixed inertial reference frame *x'y'z'* whose origin is located at the center of the Earth.



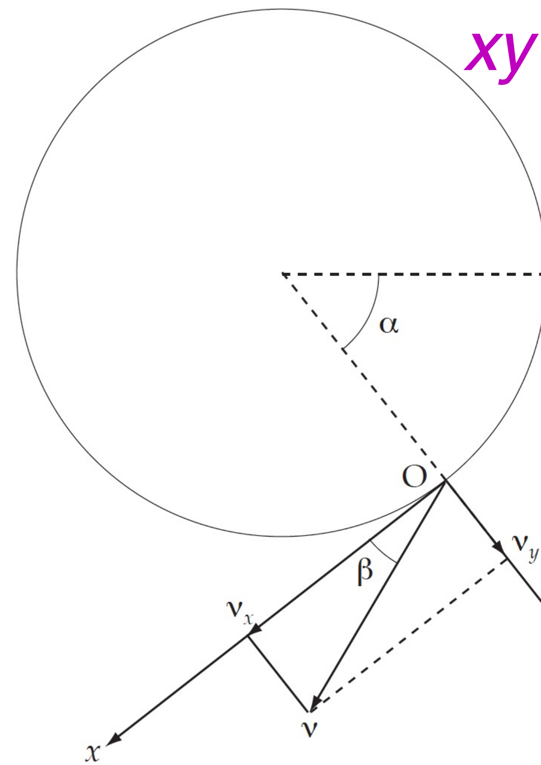
Examples of rotating coordinate systems.

ω directed along z axis.



z and z' parallel. xy plane parallel to $x'y'$ plane.

ω directed in xy plane.

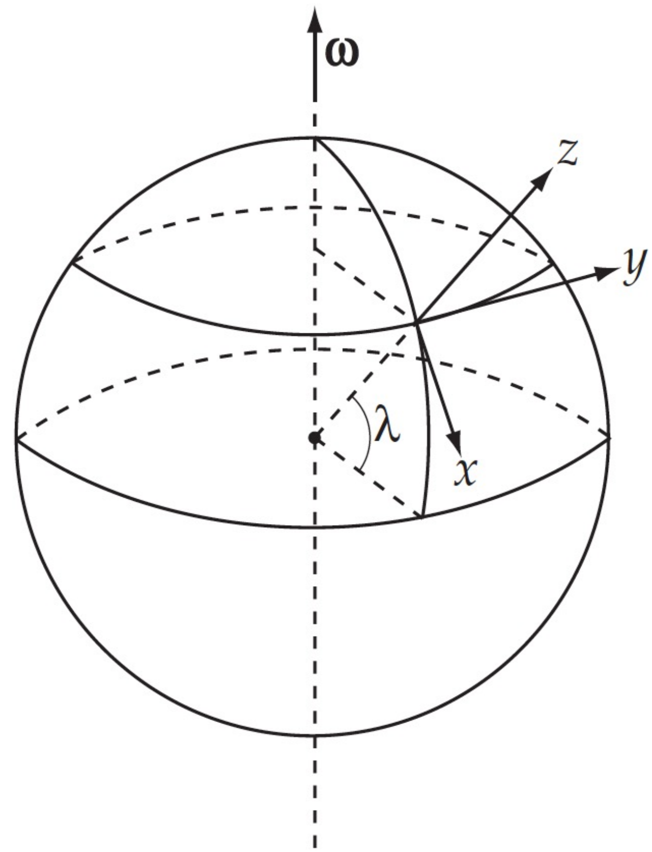


z parallel to $x'y'$ plane. y directed radially. xy plane tangential to surface.

Problem 10.8.

If a particle is projected vertically upward to a height h above the Earth's surface at a northern latitude λ, how far from its launch position does it hit the ground?

Neglect air resistance and consider only small vertical heights.



ENOUGH FOR TODAY?