Useful Relations

$$\bar{A} \bullet \bar{B} = A_x B_x + A_y B_y + A_z B_z$$

$$\bar{C} = \bar{A} \times \bar{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\bar{A} \bullet \left(\bar{B} \times \bar{C}\right) = \bar{B} \bullet \left(\bar{C} \times \bar{A}\right) = \bar{C} \bullet \left(\bar{A} \times \bar{B}\right)$$

$$\bar{A} \times \left(\bar{B} \times \bar{C}\right) = \left(\bar{A} \bullet \bar{C}\right)\bar{B} - \left(\bar{A} \bullet \bar{B}\right)\bar{C}$$

$$\left(\bar{A} \times \bar{B}\right) \bullet \left(\bar{C} \times \bar{D}\right) = \bar{A} \bullet \left[\bar{B} \times \left(\bar{C} \times \bar{D}\right)\right]$$

$$\left(\bar{A} \times \bar{B}\right) \times \left(\bar{C} \times \bar{D}\right) = \left[\left(\bar{A} \times \bar{B}\right) \bullet \bar{D}\right]\bar{C} - \left[\left(\bar{A} \times \bar{B}\right) \bullet \bar{C}\right]\bar{D}$$

Linear momentum:

$$\vec{p} = m\vec{v}$$

Angular momentum:

$$\vec{L} = \vec{r} \times \vec{p}$$

Work done by a force:

$$W = \int_1^2 \vec{F} \bullet d\vec{r}$$

Relation between force and potential energy:

$$\vec{F} = -\vec{\nabla} U$$

Simple Harmonic Oscillator:

$$\ddot{x} + \omega_0^2 x = 0$$

Damped Harmonic Oscillator:

$$\ddot{x} + \frac{b}{m}\dot{x} + \omega_0^2 x = 0$$

Driven harmonic motion:

$$\ddot{x} + \frac{b}{m}\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

Van der Pol equation:

$$\ddot{x} + \mu\left(x^2 - a^2\right)\dot{x} + \omega_0^2 x = 0$$

Lyapunov exponent:

$$\lambda = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\ln\left|\frac{df\left(x_i\right)}{dx}\right|$$

Gravitational force:

$$\vec{F} = -G\frac{mM}{r^2}\hat{r}$$

Gravitational field:

$$\vec{g} = -\vec{\nabla}\Phi$$

Gravitational potential:

$$\Phi = -G\frac{M}{r}$$

Gravitational potential energy:

$$U = m\Phi$$

Poisson's equation:

$$\nabla^2\Phi = 4\pi G\rho$$

Euler's equations without external constraints:

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right) = 0$$

Euler's equations with external constraints $g\{y; x\} = 0$:

$$\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right) + \lambda(x)\left(\frac{\partial g}{\partial y}\right) = 0$$

where $\lambda(x)$ is the Lagrange undetermined multiplier.

Hamilton's principle:

$$\delta \int_{t_1}^{t_2} (T - U)\, dt = 0$$

The Lagrange equations of motion:

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0$$

Lagrange equations of motion with undetermined multipliers:

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \sum_{k=1}^{m} \lambda_k(t)\frac{\partial f_k}{\partial q_j} = 0$$

Definition of the Hamiltonian:

$$H = \sum_j \left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) - L$$

Hamilton's equations of motion:

$$\frac{\partial H}{\partial p_j} - \dot{q}_j = 0$$

$$\frac{\partial H}{\partial q_j} + \dot{p}_j = 0$$

Orbital motion:

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\frac{l^2}{\mu r^2} + U(r)$$

$$\frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}(E-U) - \frac{l^2}{\mu^2 r^2}}$$

$$\theta(r) = \pm\int \frac{\left(\frac{l}{\mu r^2}\right)}{\sqrt{\frac{2}{\mu}(E-U) - \frac{l^2}{\mu^2 r^2}}} dr$$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{l^2} F(r)$$

Effective potential energy:

$$V(r) = U(r) + \frac{l^2}{2\mu r^2}$$

Center of Mass:

$$\bar{R}_{cm} = \frac{\sum_i m_i \bar{r}_i}{\sum_i m_i} = \frac{1}{M}\sum_i m_i \bar{r}_i$$

Linear momentum of a system of particles:

$$\bar{P} = M\dot{\bar{R}}$$

$$\frac{d\bar{P}}{dt} = \bar{F}_{ext}$$

Angular momentum of a system of particles:

$$\bar{L} = \bar{L}_{cm} + \bar{L}_{wrt,cm}$$

$$\frac{d\bar{L}}{dt} = \sum_i \left\{ \bar{r}_i \times \bar{F}_{i,ext} \right\}$$

Scattering cross sections:

$$\sigma(\theta) = \frac{\text{\#of interactions per target nucleus into an area d}\Omega'\text{ at }\theta}{\text{\# of incident particles per units area}} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Rutherford scattering:

$$\sigma(\theta) = \left(\frac{k}{4T_0'}\right)^2 \frac{1}{\sin^4(\theta/2)}$$

Rocket equations:

$$R\,u = M\,a$$

$$v_f = v_i + u\ln\left(\frac{M_i}{M_f}\right)$$

Fixed and rotating coordinate systems:

$$v_f = \left(\frac{d\bar{r}'}{dt}\right)_{fixed} = \left(\frac{d\bar{R}}{dt}\right)_{fixed} + \left(\frac{d\bar{r}}{dt}\right)_{rotating} + \bar{\omega}\times\bar{r} = V + v_r + \bar{\omega}\times\bar{r}$$

Effective forces in rotating coordinate systems:

$$\bar{F}_{eff} = m\bar{a}_f - 2m\bar{\omega}\times\bar{v}_r - m\bar{\omega}\times\left\{\bar{\omega}\times\bar{r}\right\}$$

Kinetic energy of a rigid object:

$$T = \frac{1}{2}MV^2 + \frac{1}{2}\sum_{i,j} I_{ij}\omega_i\omega_j = T_{CM} + T_{rot}$$

where the quantity I_{ij} is called the inertia tensor, and is a 3 x 3 matrix:

$$\{I\} = \begin{pmatrix} \sum_\alpha m_\alpha\left(r_{\alpha,2}{}^2 + r_{\alpha,3}{}^2\right) & -\sum_\alpha m_\alpha r_{\alpha,1} r_{\alpha,2} & -\sum_\alpha m_\alpha r_{\alpha,1} r_{\alpha,3} \\ -\sum_\alpha m_\alpha r_{\alpha,2} r_{\alpha,1} & \sum_\alpha m_\alpha\left(r_{\alpha,1}{}^2 + r_{\alpha,3}{}^2\right) & -\sum_\alpha m_\alpha r_{\alpha,2} r_{\alpha,3} \\ -\sum_\alpha m_\alpha r_{\alpha,3} r_{\alpha,1} & -\sum_\alpha m_\alpha r_{\alpha,3} r_{\alpha,2} & \sum_\alpha m_\alpha\left(r_{\alpha,1}{}^2 + r_{\alpha,2}{}^2\right) \end{pmatrix}$$

Transformation of the inertia tensors for rotations:

$$\{I'\} = \{\lambda\}\{I\}\{\lambda^t\}$$

The angular momentum of a rotating rigid body is equal to

$$L_i = \sum_k I_{ik}\omega_k$$

The Euler equations in a force-free field:

$$(I_1 - I_2)\omega_1\omega_2 - I_3\dot{\omega}_3 = 0$$

$$(I_2 - I_3)\omega_2\omega_3 - I_1\dot{\omega}_1 = 0$$

$$(I_3 - I_1)\omega_3\omega_1 - I_2\dot{\omega}_2 = 0$$

The Euler equations in a force field:

$$N_1 = I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3$$

$$N_2 = I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1$$

$$N_3 = I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2$$

Kinetic energy of a system of oscillators:

$$T = \frac{1}{2}\sum_{j,k} m_{jk}\dot{q}_j\dot{q}_k$$

Potential energy of a system of oscillators:

$$U\left(q_1, q_2, \ldots\right) = \frac{1}{2}\sum_{j,k} A_{jk} q_j q_k$$

where

$$A_{jk} = \left.\frac{\partial^2 U}{\partial q_j\, \partial q_k}\right|_0$$

Equations of motion of coupled oscillators:

$$\frac{1}{2}\sum_{k}\left(A_{kj} q_k + m_{kj}\ddot{q}_k\right) = 0$$

Solution of the loaded string:

$$q_j\left(t\right) = \sum_{s}\beta_s a_s \sin\left(j\frac{s\pi}{\left(n+1\right)}\right)e^{i\omega_s t}$$

where

$$\omega_s = 2\sqrt{\frac{\tau}{md}}\sin\left(\frac{s\pi}{2\left(n+1\right)}\right)$$

Solution of the continuous string:

$$q_j(t) = \sum_s \beta_s e^{i\omega_s t} \sin\left(s\pi \frac{x}{L} \right) = q(x,t)$$

where

$$\omega_s = s\frac{\pi}{L}\sqrt{\frac{\tau}{\rho}}$$

and

$$\beta_s = \mu_s + i\nu_s$$

Kinetic and potential energy of the string:

$$T = \frac{\rho L}{4}\sum_s \omega_s{}^2 \left(\mu_s \sin(\omega_s t) + \nu_s \cos(\omega_s t) \right)^2$$

$$U = \frac{\rho L}{4}\sum_s \omega_s{}^2 \left(\mu_s \cos(\omega_s t) - \nu_s \sin(\omega_s t) \right)^2$$

The ideal wave equation:

$$\frac{\partial^2 q}{\partial x^2} = \frac{\rho}{\tau}\frac{\partial^2 q}{\partial t^2}$$

The real wave equation:

$$\tau\frac{\partial^2 q}{\partial x^2} - D\frac{\partial q}{\partial t} + F(x,t) = \rho\frac{\partial^2 q}{\partial t^2}$$

The phase velocity of a wave:

$$V = \frac{\omega}{k} = \sqrt{\frac{\tau d}{m}}\frac{\sin\left\{\frac{kd}{2}\right\}}{\frac{kd}{2}}$$

The wavelength of a wave:

$$\lambda = \frac{2\pi}{k}$$

Important additional information

Best baseball team in the USA:

Yankees

Best soccer team in the world:

AJAX

Best airline in the world:

KLM (Koninklijke Luchtvaart Maatschappij = Royal Dutch Airlines)

If in doubt, the correct answer may be:

Yankees, AJAX, the Netherlands, or KLM.

APPENDIX D
Useful Formulas*

D.1 Binomial Expansion

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + \binom{n}{r}x^r + \cdots, \quad |x| < 1 \tag{D.1}$$

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + (-1)^r\binom{n}{r}x^r + \cdots, \quad |x| < 1 \tag{D.2}$$

where the **binomial coefficient** is

$$\binom{n}{r} \equiv \frac{n!}{(n-r)!\,r!} \tag{D.3}$$

Some particularly useful cases of the above are

$$(1 \pm x)^{1/2} = 1 \pm \frac{1}{2}x - \frac{1}{8}x^2 \pm \frac{1}{16}x^3 - \cdots \tag{D.4}$$

$$(1 \pm x)^{1/3} = 1 \pm \frac{1}{3}x - \frac{1}{9}x^2 \pm \frac{5}{81}x^3 - \cdots \tag{D.5}$$

*An extensive list may be found, for example, in Dwight (Dw61).

$$(1 \pm x)^{-1/2} = 1 \mp \frac{1}{2}x + \frac{3}{8}x^2 \mp \frac{5}{16}x^3 + \cdots \tag{D.6}$$

$$(1 \pm x)^{-1/3} = 1 \mp \frac{1}{3}x + \frac{2}{9}x^2 \mp \frac{14}{81}x^3 + \cdots \tag{D.7}$$

$$(1 \pm x)^{-1} = 1 \mp x + x^2 \mp x^3 + \cdots \tag{D.8}$$

$$(1 \pm x)^{-2} = 1 \mp 2x + 3x^2 \mp 4x^3 + \cdots \tag{D.9}$$

$$(1 \pm x)^{-3} = 1 \mp 3x + 6x^2 \mp 10x^3 + \cdots \tag{D.10}$$

For convergence of *all* the above series, we must have $|x| < 1$.

D.2 Trigonometric Relations

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \tag{D.11}$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \tag{D.12}$$

$$\sin 2A = 2 \sin A \cos A = \frac{2\tan A}{1 + \tan^2 A} \tag{D.13}$$

$$\cos 2A = 2\cos^2 A - 1 \tag{D.14}$$

$$\sin^2 \frac{A}{2} = \frac{1}{2}(1 - \cos A) \tag{D.15}$$

$$\cos^2 \frac{A}{2} = \frac{1}{2}(1 + \cos A) \tag{D.16}$$

$$\sin^2 A = \frac{1}{2}(1 - \cos 2A) \tag{D.17}$$

$$\sin^3 A = \frac{1}{4}(3 \sin A - \sin 3A) \tag{D.18}$$

$$\sin^4 A = \frac{1}{8}(3 - 4\cos 2A + \cos 4A) \tag{D.19}$$

$$\cos^2 A = \frac{1}{2}(1 + \cos 2A) \tag{D.20}$$

$$\cos^3 A = \frac{1}{4}(3\cos A + \cos 3A) \tag{D.21}$$

$$\cos^4 A = \frac{1}{8}(3 + 4\cos 2A + \cos 4A) \tag{D.22}$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \quad \text{(D.23)}$$

$$\tan^2 \frac{A}{2} = \frac{1 - \cos A}{1 + \cos A} \quad \text{(D.24)}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \quad \text{(D.25)}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \text{(D.26)}$$

$$e^{ix} = \cos x + i \sin x \quad \text{(D.27)}$$

D.3 Trigonometric Series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \text{(D.28)}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \text{(D.29)}$$

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots, \quad |x| < \pi/2 \quad \text{(D.30)}$$

$$\sin^{-1} x = x + \frac{x^3}{6} + \frac{3}{40}x^5 + \cdots, \quad \begin{cases} |x| < 1 \\ |\sin^{-1} x| < \pi/2 \end{cases} \quad \text{(D.31)}$$

$$\cos^{-1} x = \frac{\pi}{2} - x - \frac{x^3}{6} - \frac{3}{40}x^5 - \cdots, \quad \begin{cases} |x| < 1 \\ 0 < \cos^{-1} x < \pi \end{cases} \quad \text{(D.32)}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \quad |x| < 1 \quad \text{(D.33)}$$

D.4 Exponential and Logarithmic Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{(D.34)}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad |x| < 1, \quad x = 1 \quad \text{(D.35)}$$

$$\ln\left[\sqrt{(x^2/a^2) + 1} + (x/a)\right] = \sinh^{-1} x/a \quad \text{(D.36)}$$

$$= -\ln\left[\sqrt{(x^2/a^2) + 1} - (x/a)\right] \quad \text{(D.37)}$$

D.5 Complex Quantities

Cartesian form: $z = x + iy$, complex conjugate $z^* = x - iy$, $i = \sqrt{-1}$ (D.38)

Polar form: $z = |z| e^{i\theta}$ (D.39)

$$z^* = |z| e^{-i\theta} \quad \text{(D.40)}$$

$$zz^* = |z|^2 = x^2 + y^2 \quad \text{(D.41)}$$

Real part of z: $\mathrm{Re}\, z = \frac{1}{2}(z + z^*) = x$ (D.42)

Imaginary part of z: $\mathrm{Im}\, z = -\frac{1}{2}(z - z^*) = y$ (D.43)

Euler's formula: $e^{i\theta} = \cos\theta + i \sin\theta$ (D.44)

D.6 Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{(D.45)}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{(D.46)}$$

$$\tanh x = \frac{e^{2x} - 1}{e^{2x} + 1} \quad \text{(D.47)}$$

$$\sin ix = i \sinh x \quad \text{(D.48)}$$

$$\cos ix = \cosh x \quad \text{(D.49)}$$

$$\sinh ix = i \sin x \quad \text{(D.50)}$$

$$\cosh ix = \cos x \quad \text{(D.51)}$$

$$\sinh^{-1} x = \tanh^{-1}\left(\frac{x}{\sqrt{x^2 + 1}}\right) \quad \text{(D.52)}$$

$$= \ln(x + \sqrt{x^2 + 1}) \quad \text{(D.53)}$$

$$= \cosh^{-1}(\sqrt{x^2 + 1}), \quad \begin{cases} > 0, & x > 0 \\ < 0, & x < 0 \end{cases} \quad \text{(D.54)}$$

$$\cosh^{-1} x = \pm \tanh^{-1}\left(\frac{\sqrt{x^2 - 1}}{x}\right), \quad x > 1 \quad \text{(D.55)}$$

$$= \pm \ln(x + \sqrt{x^2 - 1}), \quad x > 1 \quad \text{(D.56)}$$

$$\cosh^{-1} x = \pm \sinh^{-1}(\sqrt{x^2 - 1}), \quad x > 1 \tag{D.57}$$

$$\frac{d}{dy}\sinh y = \cosh y \tag{D.58}$$

$$\frac{d}{dy}\cosh y = \sinh y \tag{D.59}$$

$$\sinh(x_1 + x_2) = \sinh x_1 \cosh x_2 + \cosh x_1 \sinh x_2 \tag{D.60}$$

$$\cosh(x_1 + x_2) = \cosh x_1 \cosh x_2 + \sinh x_1 \sinh x_2 \tag{D.61}$$

$$\cosh^2 x - \sinh^2 x = 1 \tag{D.62}$$

PROBLEMS

D-1. Is it possible to ascribe a meaning to the inequality $z_1 < z_2$? Explain. Does the inequality $|z_1| < |z_2|$ have a different meaning?

D-2. Solve the following equations:
(a) $z^2 + 2z + 2 = 0$ **(b)** $2z^2 + z + 2 = 0$

D-3. Express the following in polar form:
(a) $z_1 = i$ **(b)** $z_2 = -1$
(c) $z_3 = 1 + i\sqrt{3}$ **(d)** $z_4 = 1 + 2i$
(e) Find the product $z_1 z_2$ **(f)** Find the product $z_1 z_3$
(g) Find the product $z_3 z_4$

D-4. Express $(z^2 - 1)^{-1/2}$ in polar form.

D-5. If the function $w = \sin^{-1} z$ is defined as the inverse of $z = \sin w$, then use the Euler relation for $\sin w$ to find an equation for $\exp(iw)$. Solve this equation and obtain the result

$$w = \sin^{-1} z = -i \ln\left(iz + \sqrt{1 - z^2}\right)$$

D-6. Show that

$$y = Ae^{ix} + Be^{-ix}$$

can be written as

$$y = C\cos(x - \delta)$$

where A and B are *complex* but where C and δ are *real*.

D-7. Show that
(a) $\sinh(x_1 + x_2) = \sinh x_1 \cosh x_2 + \cosh x_1 \sinh x_2$
(b) $\cosh(x_1 + x_2) = \cosh x_1 \cosh x_2 + \sinh x_1 \sinh x_2$

APPENDIX E

*Useful Integrals**

E.1 Algebraic Functions

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right), \quad \left|\tan^{-1}\left(\frac{x}{a}\right)\right| < \frac{\pi}{2} \tag{E.1}$$

$$\int \frac{x\,dx}{a^2 + x^2} = \frac{1}{2}\ln(a^2 + x^2) \tag{E.2}$$

$$\int \frac{dx}{x(a^2 + x^2)} = \frac{1}{2a^2}\ln\left(\frac{x^2}{a^2 + x^2}\right) \tag{E.3}$$

$$\int \frac{dx}{a^2x^2 - b^2} = \frac{1}{2ab}\ln\left(\frac{ax - b}{ax + b}\right) \tag{E.4a}$$

$$= -\frac{1}{ab}\coth^{-1}\left(\frac{ax}{b}\right), \quad a^2x^2 > b^2 \tag{E.4b}$$

$$= -\frac{1}{ab}\tanh^{-1}\left(\frac{ax}{b}\right), \quad a^2x^2 < b^2 \tag{E.4c}$$

$$\int \frac{dx}{\sqrt{a + bx}} = \frac{2}{b}\sqrt{a + bx} \tag{E.5}$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln(x + \sqrt{x^2 + a^2}) \tag{E.6}$$

*This list is confined to those (nontrivial) integrals that arise in the text and in the problems. Extremely useful compilations are, for example, Pierce and Foster (Pi57) and Dwight (Dw61).

$$\int \frac{x^2dx}{\sqrt{a^2-x^2}} = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \tag{E.7}$$

$$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}}\ln(2\sqrt{a}\sqrt{ax^2+bx+c}+2ax+b), \quad a>0 \tag{E.8a}$$

$$= \frac{1}{\sqrt{a}}\sinh^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right), \quad \begin{cases} a>0 \\ 4ac>b^2 \end{cases} \tag{E.8b}$$

$$= -\frac{1}{\sqrt{-a}}\sin^{-1}\left(\frac{2ax+b}{\sqrt{b^2-4ac}}\right), \quad \begin{cases} a<0 \\ b^2>4ac \\ |2ax+b|<\sqrt{b^2-4ac} \end{cases} \tag{E.8c}$$

$$\int \frac{x\,dx}{\sqrt{ax^2+bx+c}} = \frac{1}{a}\sqrt{ax^2+bx+c} - \frac{b}{2a}\int\frac{dx}{\sqrt{ax^2+bx+c}} \tag{E.9}$$

$$\int \frac{dx}{x\sqrt{ax^2+bx+c}} = -\frac{1}{\sqrt{c}}\sinh^{-1}\left(\frac{bx+2c}{|x|\sqrt{4ac-b^2}}\right), \quad \begin{cases} c>0 \\ 4ac>b^2 \end{cases} \tag{E.10a}$$

$$= \frac{1}{\sqrt{-c}}\sin^{-1}\left(\frac{bx+2c}{|x|\sqrt{b^2-4ac}}\right), \quad \begin{cases} c<0 \\ b^2>4ac \end{cases} \tag{E.10b}$$

$$= -\frac{1}{\sqrt{c}}\ln\left(\frac{2\sqrt{c}}{x}\sqrt{ax^2+bx+c}+\frac{2c}{x}+b\right), \quad c>0 \tag{E.10c}$$

$$\int \sqrt{ax^2+bx+c}\,dx = \frac{2ax+b}{4a}\sqrt{ax^2+bx+c} + \frac{4ac-b^2}{8a}\int\frac{dx}{\sqrt{ax^2+bx+c}} \tag{E.11}$$

E.2 Trigonometric Functions

$$\int \sin^2 x\,dx = \frac{x}{2} - \frac{1}{4}\sin 2x \tag{E.12}$$

$$\int \cos^2 x\,dx = \frac{x}{2} + \frac{1}{4}\sin 2x \tag{E.13}$$

$$\int \frac{dx}{a+b\sin x} = \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left[\frac{a\tan(x/2)+b}{\sqrt{a^2-b^2}}\right], \quad a^2>b^2 \tag{E.14}$$

$$\int \frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left[\frac{(a-b)\tan(x/2)}{\sqrt{a^2-b^2}}\right], \quad a^2>b^2 \tag{E.15}$$

$$\int \frac{dx}{(a+b\cos x)^2} = \frac{b\sin x}{(b^2-a^2)(a+b\cos x)} - \frac{a}{b^2-a^2}\int\frac{dx}{a+b\cos x} \tag{E.16}$$

$$\int \tan x\,dx = -\ln|\cos x| \tag{E.17a}$$

$$\int \tanh x\,dx = \ln\cosh x \tag{E.17b}$$

$$\int e^{ax}\sin x\,dx = \frac{e^{ax}}{a^2+1}(a\sin x - \cos x) \tag{E.18a}$$

$$\int e^{ax}\sin^2 x\,dx = \frac{e^{ax}}{a^2+4}\left(a\sin^2 x - 2\sin x\cos x + \frac{2}{a}\right) \tag{E.18b}$$

$$\int_{-\infty}^{\infty} e^{-ax^2}dx = \sqrt{\pi/a} \tag{E.18c}$$

E.3 Gamma Functions

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx \tag{E.19a}$$

$$= \int_0^1 [\ln(1/x)]^{n-1}dx \tag{E.19b}$$

$$\Gamma(n) = (n-1)!, \quad \text{for } n = \text{positive integer} \tag{E.19c}$$

$$n\Gamma(n) = \Gamma(n+1) \tag{E.20}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{E.21}$$

$$\Gamma(1) = 1 \tag{E.22}$$

$$\Gamma\left(1\frac{1}{4}\right) = 0.906 \tag{E.23}$$

$$\Gamma\left(1\frac{3}{4}\right) = 0.919 \tag{E.24}$$

$$\Gamma(2) = 1 \tag{E.25}$$

$$\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\sqrt{\pi}}{n} \frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)} \tag{E.26}$$

$$\int_0^1 x^m (1-x^2)^n \, dx = \frac{\Gamma(n+1)\Gamma\left(\frac{m+1}{2}\right)}{2\Gamma\left(n+\frac{m+3}{2}\right)} \tag{E.27a}$$

$$\int_0^{\pi/2} \cos^n x \, dx = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}, \quad n > -1 \tag{E.27b}$$

APPENDIX **F**

Differential Relations in Different Coordinate Systems

F.1 Rectangular Coordinates

$$\textbf{grad}\, U = \nabla U = \sum_i \mathbf{e}_i \frac{\partial U}{\partial x_i} \tag{F.1}$$

$$\text{div}\, \mathbf{A} = \nabla \cdot \mathbf{A} = \sum_i \frac{\partial A_i}{\partial x_i} \tag{F.2}$$

$$\textbf{curl}\, \mathbf{A} = \nabla \times \mathbf{A} = \sum_{i,j,k} \varepsilon_{ijk} \frac{\partial A_k}{\partial x_j} \mathbf{e}_i \tag{F.3}$$

$$\nabla^2 U = \nabla \cdot \nabla U = \sum_i \frac{\partial^2 U}{\partial x_i^2} \tag{F.4}$$

F.2 Cylindrical Coordinates

Refer to Figures F-1 and F-2.

$$x_1 = r\cos\phi, \quad x_2 = r\sin\phi, \quad x_3 = z \tag{F.5}$$

$$r = \sqrt{x_1^2 + x_2^2}, \quad \phi = \tan^{-1}\frac{x_2}{x_1}, \quad z = x_3 \tag{F.6}$$

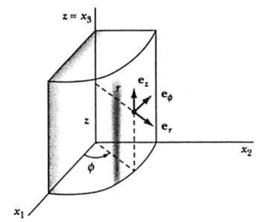

FIGURE F-1

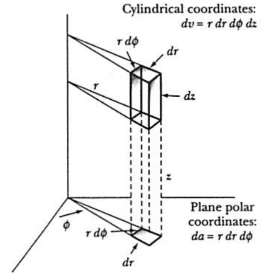

FIGURE F-2

$$ds^2 = dr^2 + r^2\,d\phi^2 + dz^2 \tag{F.7}$$

$$dv = r\,dr\,d\phi\,dz \tag{F.8}$$

$$\textbf{grad}\ \psi = \nabla\psi = \mathbf{e}_r \frac{\partial\psi}{\partial r} + \mathbf{e}_\phi \frac{1}{r}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_z \frac{\partial\psi}{\partial z} \tag{F.9}$$

$$\text{div}\ \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z} \tag{F.10}$$

$$\textbf{curl A} = \mathbf{e}_r\left(\frac{1}{r}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right) + \mathbf{e}_\phi\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \mathbf{e}_z\left(\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi) - \frac{1}{r}\frac{\partial A_r}{\partial\phi}\right) \tag{F.11}$$

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} \tag{F.12}$$

F.3 Spherical Coordinates

Refer to Figures F-3 and F-4

$$x_1 = r\sin\theta\cos\phi, \quad x_2 = r\sin\theta\sin\phi, \quad x_3 = r\cos\theta \tag{F.13}$$

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad \theta = \cos^{-1}\frac{x_3}{r}, \quad \phi = \tan^{-1}\frac{x_2}{x_1} \tag{F.14}$$

$$ds^2 = dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2 \tag{F.15}$$

$$dv = r^2\sin\theta\,dr\,d\theta\,d\phi \tag{F.16}$$

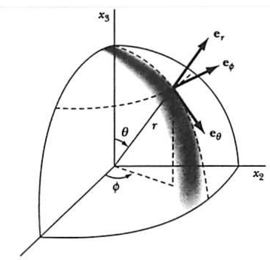

FIGURE F-3

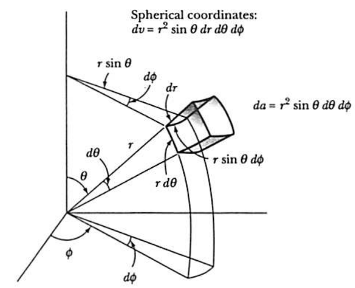

FIGURE F-4

$$\mathbf{grad}\,\psi = \nabla\psi = \mathbf{e}_r\frac{\partial\psi}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_\phi\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi} \tag{F.17}$$

$$\mathrm{div}\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta} + \frac{\partial A_\phi}{\partial\phi} \tag{F.18}$$

$$\begin{aligned}\mathbf{curl\,A} &= \mathbf{e}_r\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right]\\ &\quad + \mathbf{e}_\theta\frac{1}{r\sin\theta}\left[\frac{\partial A_r}{\partial\phi} - \sin\theta\frac{\partial}{\partial r}(rA_\phi)\right] + \mathbf{e}_\phi\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\end{aligned} \tag{F.19}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} \tag{F.20}$$