

Chapter 7. Electrodynamics

7.1. Electromotive Force

An electric current is flowing when the electric charges are in motion. In order to sustain an electric current we have to apply a force on these charges. In most materials the current density \vec{J} is proportional to the force per unit charge:

$$\vec{J} = \sigma \vec{f}$$

The constant of proportionality σ is called the **conductivity** of the material. Instead of specifying the conductivity, it is more common to specify the **resistivity** ρ :

$$\rho = \frac{1}{\sigma}$$

For conductors the resistivity is typically $10^{-8} \Omega\text{-m}$; for semiconductor it varies between $0.01 \Omega\text{-m}$ and $1 \Omega\text{-m}$, and for insulators it varies between $10^5 \Omega\text{-m}$ and $10^6 \Omega\text{-m}$. In most cases the force on the charges is the electromagnetic force. In that case the current density is equal to:

$$\vec{J} = \sigma (\vec{E} + \vec{v} \times \vec{B})$$

If the velocity of the charges is small the second term can be ignored, and the equation for \vec{J} reduces to **Ohm's Law**:

$$\vec{J} = \sigma \vec{E}$$

Consider a wire of cross-sectional area A and length L . If a potential difference V is applied between the ends of the wire, it will produce an electric field inside the wire of magnitude

$$E = \frac{V}{L}$$

The current density in the wire is therefore equal to

$$J = \sigma \frac{V}{L}$$

The total current flowing through the wire is therefore equal to

$$I = JA = \sigma A \frac{V}{L}$$

This equation shows that the current flowing from one electrode to the other electrode is proportional to the potential difference between them. This is a rather surprising result since the charge carriers are constantly accelerating. However, the proportionality between the current and the potential difference has been found to be correct for most materials. This relation can be written as

$$V = IR$$

The constant of proportionality R is called the **resistance** of the material. It is in general a function of the geometry of the system and the conductivity of the materials between the electrodes. The unit of resistance is the **ohm** (Ω). The resistance of the wire is equal to

$$R = \frac{V}{I} = \frac{V}{\sigma A \frac{V}{L}} = \frac{1}{\sigma} \frac{L}{A} = \rho \frac{L}{A}$$

To create a current we have to do work. The work required to move a unit of charge across a potential difference V is equal to V . To establish a current I , we need to deliver a power P where

$$P = VI = I^2R$$

The unit of power is the **Watt** ($1 \text{ W} = 1 \text{ J/s}$). The work done by the electric force on the charge carriers is converted into heat (**Joule heating**).

Example: Problem 7.1

Two concentric metal spherical shells, of radius a and b , respectively, are separated by weakly conducting material of conductivity σ .

- a) If they are maintained at a potential difference V , what current flows from one to the other?
- b) What is the resistance between the shells?

- a) Suppose a charge Q is placed on the inner shell. The electric field in the region between the shells will be equal to

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{r}$$

The corresponding potential difference between the spheres is equal to

$$V_a - V_b = -\int_b^a \vec{E} \cdot d\vec{r} = \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{a} - \frac{1}{b} \right)$$

Therefore, in order to maintain a potential difference V between the spheres, we must place a charge Q equal to

$$Q = \frac{4\pi\epsilon_0 V}{\left(\frac{1}{a} - \frac{1}{b} \right)}$$

on the center shell. The total current flowing between the two shells is equal to

$$I = \oint_{\text{Sphere}} \vec{J} \cdot d\vec{a} = \sigma \oint_{\text{Sphere}} \vec{E} \cdot d\vec{a} = \sigma \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} 4\pi r^2 = \sigma \frac{Q}{\epsilon_0} = 4\pi\sigma \frac{V}{\left(\frac{1}{a} - \frac{1}{b} \right)}$$

b) The resistance between the shells can be obtained from Ohm's law:

$$R = \frac{V}{I} = \frac{V}{4\pi\sigma \frac{V}{\left(\frac{1}{a} - \frac{1}{b} \right)}} = \frac{1}{4\pi\sigma} \left(\frac{1}{a} - \frac{1}{b} \right)$$

Example: Problem 7.2

a) Two metal objects are embedded in weakly conducting material of conductivity σ (see Figure 7.1). Show that the resistance between them is related to the capacitance of the arrangement by

$$R = \frac{\epsilon_0}{\sigma C}$$

b) Suppose you connected a battery between 1 and 2 and charged them up to a potential difference V_0 . If you then disconnect the battery, the charge will gradually leak off. Show that $V(t) = V_0 \exp(-t/\tau)$, and find the time constant τ in terms of ϵ_0 and σ .

a) Suppose a charge Q is placed on the positively charged conductor. The current flowing from the positively charged conductor is equal to

$$I = \oint_{\text{Surface}} \vec{J} \cdot d\vec{a}$$

where the surface integral is taken over a surface that encloses the positively charged conductor (for example, the dashed surface in Figure 7.1). The expression for I can be rewritten in terms of the electric field as

$$I = \sigma \oint_{\text{Surface}} \vec{E} \cdot d\vec{a}$$

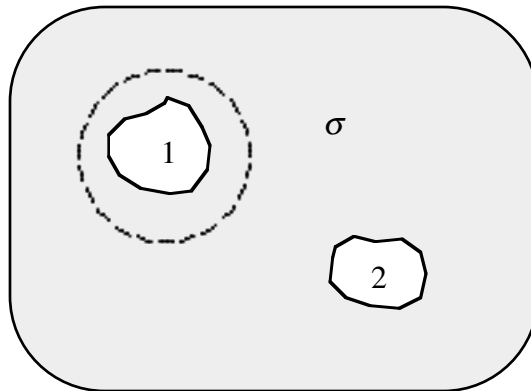


Figure 7.1. Problem 7.2.

Using Gauss's law to express the surface integral of \vec{E} in terms of the total enclosed charge we obtain

$$I = \sigma \frac{Q}{\epsilon_0}$$

The charge on the conductor is related to the capacitance of the arrangement and the potential difference between the conductors:

$$Q = CV$$

The current I is therefore equal to

$$I = \frac{\sigma}{\epsilon_0} CV$$

The resistance of the system can be calculated using Ohm's law:

$$R = \frac{V}{I} = \frac{V}{\frac{\sigma}{\epsilon_0} CV} = \frac{\epsilon_0}{\sigma C}$$

b) The charge Q residing on the positively charged conductor is equal to

$$Q = CV = CRI = -CR \frac{dQ}{dt}$$

This equation can be rewritten as

$$\frac{dQ}{dt} + \frac{1}{CR} Q = 0$$

and has the following solution:

$$Q(t) = Q_0 e^{-t/RC}$$

The potential difference V is equal to

$$V(t) = \frac{Q(t)}{C} = \frac{Q_0}{C} e^{-t/RC} = V_0 e^{-t/RC}$$

The decay constant τ is equal to

$$\tau = RC = \frac{\epsilon_0}{\sigma}$$

In any electric circuit a current will only exist if a driving force is available. The most common sources of the driving force are batteries and generators. When a circuit is hooked up to a power source a current will start to flow. In a single-loop circuit the current will be the same everywhere. Consider the situation in which the currents are not the same (see Figure 7.2). If $I_{\text{in}} > I_{\text{out}}$ then positive charge will accumulate in the middle. This accumulation of positive charge will generate an electric field (see Figure 7.2) that slows down the incoming charges and speeds up the outgoing charges. A reduction in the velocity of the incoming charges will reduce the incoming current. An increase in the velocity of the outgoing charges will increase the outgoing current. The current will change until $I_{\text{in}} = I_{\text{out}}$.

The total force f on the charge carriers (per unit charge) is equal to the sum of the source force, f_s , and the electric force:

$$\vec{f} = \vec{f}_s + \vec{E}$$

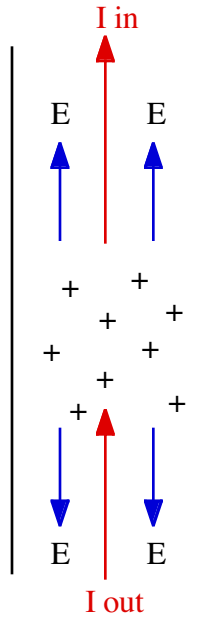


Figure 7.2. Current flow.

The work required to move one unit of charge once around the circuit is equal to

$$\oint \vec{f} \cdot d\vec{l} = \oint \vec{f}_s \cdot d\vec{l} + \oint \vec{E} \cdot d\vec{l} = \oint \vec{f}_s \cdot d\vec{l} = \varepsilon$$

where ε is called the **electromotive force** or **emf**. The emf determines the current flowing through the circuit. This can be most easily seen by rewriting the force \vec{f} on the charge carriers in terms of the current density \vec{J}

$$\varepsilon = \oint \vec{f} \cdot d\vec{l} = \oint \frac{\vec{J}}{\sigma} \cdot d\vec{l} = \oint \frac{I}{a\sigma} dl = I \oint \frac{dl}{a\sigma} = IR$$

Here, a is the cross-sectional area of the wire (perpendicular to the direction of the current).

Example: Problem 7.5

- a) Show that electrostatic force alone cannot be used to drive current around a circuit.
- b) A rectangular loop of wire is situated so that one end is between the plates of a parallel-plate capacitor (see Figure 7.3), oriented parallel to the field $E = \sigma/\varepsilon_0$. The other end is way outside, where the field is essentially zero. If the width of the loop is h and its total resistance is R , what current flows? Explain.

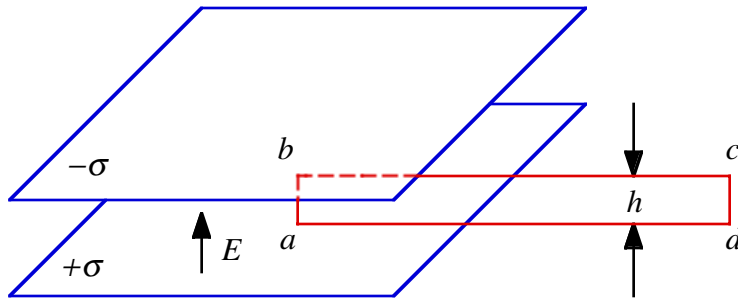


Figure 7.3. Problem 7.5.

a) If only electrostatic forces are present then the force per unit charge is equal to the electrostatic force:

$$\vec{f} = \vec{E}$$

The associated emf is therefore equal to

$$\varepsilon = \oint \vec{f} \cdot d\vec{l} = \oint \vec{E} \cdot d\vec{l} = 0$$

for any electrostatic field.

b) The only force on the charge carriers in the wire loop is the electric force. However, in part a) we concluded that the emf associated with an electric force, generated by an electrostatic field, is equal to zero. Therefore, the emf in the wire loop is equal to zero, and consequently the current in the loop is also equal to zero. Note: at first sight it might appear that there is a net emf, if we assume that the electric field generated by the capacitor is that of an ideal capacitor (that is a homogeneous field inside and no field outside). Under that assumption, the emf is equal to

$$\varepsilon = \oint \vec{E} \cdot d\vec{l} = \int_a^b \vec{E} \cdot d\vec{l} = \frac{\sigma}{\varepsilon_0} h$$

The contribution of the path integral from c to d is equal to zero since the electric field is zero there, and the contribution of the path integrals between b and c and between a and d is equal to zero since the electric field and the displacement are perpendicular there. Clearly the calculated emf is non-zero, and disagrees with the result of part a). The disagreement is a result of our incorrect assumption that the electric field outside the capacitor is equal to zero (there are fringing fields).

An important source of emf is the generator. In these devices the emf arises from the motion of a conducting wire through a magnetic field. Consider the system shown in Figure 7.4 (note:

the magnetic field is only present in the region left of the dashed line). Consider the free charges on the conductor. Since it is moving with a velocity v in a magnetic field it will experience a magnetic force. The force on a positive charge q located on segment ab of the wire loop is equal to

$$F_q = qvB$$

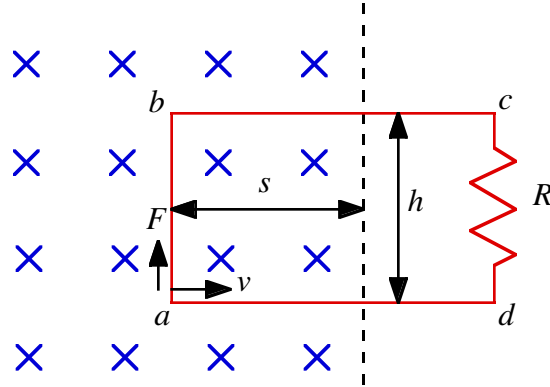


Figure 7.4. The generator.

The magnetic force per unit charge is therefore equal to

$$f_{mag} = \frac{F_q}{q} = vB$$

Since there are no other forces acting on the charges, the emf generated will be entirely due to this magnetic force. The emf will be equal to

$$\mathcal{E} = \oint \vec{f}_{mag} \cdot d\vec{l} = \int_a^b \vec{f}_{mag} \cdot d\vec{l} = vBh$$

The magnetic flux intercepted by the wire loop is equal to

$$\Phi = Bhs$$

The rate of change of the magnetic flux is equal to

$$\frac{d\Phi}{dt} = Bh \frac{ds}{dt} = -Bhv$$

Comparing the rate of change of enclosed magnetic flux and the induced emf we can conclude that

$$\varepsilon = -\frac{d\Phi}{dt}$$

This relation is called **the flux rule for motional emf**.

7.2. Faraday's Law

When a conducting wire moves in a constant magnetic field an emf is generated equal to

$$\varepsilon = -\frac{d\Phi}{dt}$$

In this case, the magnetic force is responsible for the emf. However, the same emf is generated when the wire is stationary and the magnetic field is moving. In this case, the magnetic force does not play a role (since $v = 0$) and an electric field is responsible for the emf. This electric field is not an electrostatic field (since electrostatic fields can not generate an emf; see Problem 7.5) but is induced by the changing magnetic field. The line integral of this electric field is equal to

$$\oint_{Line} \vec{E} \cdot d\vec{l} = \varepsilon = -\frac{d\Phi}{dt}$$

This equation can be rewritten by applying Stoke's theorem:

$$\oint_{Line} \vec{E} \cdot d\vec{l} = \int_{Surface} (\nabla \times \vec{E}) \cdot d\vec{a} = -\frac{d}{dt} \int_{Surface} \vec{B} \cdot d\vec{a} = - \int_{Surface} \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a}$$

Since we have not made any assumption about the surface, this equation can only be true if

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

This relation is called **Faraday's law in differential form**. The direction of the currents generated by the changing magnetic field can be obtained most easily using **Lenz's law** which states that

“ If a current flows, it will be in such a direction that the magnetic field it produces tends to counteract the change in flux that induced the emf. “

Example: Problem 7.14

A long solenoid of radius a , carrying N turns per unit length, is looped by a wire of resistance R (see Figure 7.5).

a) If the current in the solenoid is increasing,

$$\frac{dI}{dt} = k = \text{constant}$$

what current flows in the loop, and which way (left or right) does it pass through the resistor.

b) If the current I in the solenoid is constant but the solenoid is pulled out of the loop and reinserted in the opposite direction what total charge passes through the resistor?

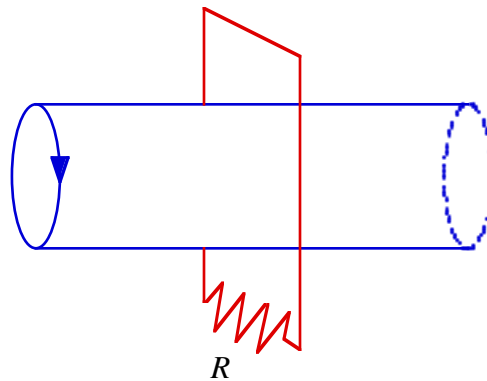


Figure 7.5. Problem 7.14.

a) Assume that the solenoid is an ideal solenoid; that is

$$\vec{B} = \mu_0 N I \hat{k}$$

If the current in the solenoid increases, the strength of the magnetic field also increases. The rate of change in the strength of the magnetic field is equal to

$$\frac{d\vec{B}}{dt} = \mu_0 N \frac{dI}{dt} \hat{k} = \mu_0 N k \hat{k}$$

The magnetic flux intercepted by the wire loop is equal to

$$\Phi = \pi a^2 B$$

The corresponding rate of change of the magnetic flux is equal to

$$\frac{d\Phi}{dt} = \pi a^2 \frac{dB}{dt} = \pi a^2 \mu_0 N k$$

The induced emf can be obtained from the flux law:

$$\varepsilon = -\frac{d\Phi}{dt} = -\pi a^2 \mu_0 N k$$

The current induced in the wire loop is equal to

$$I = \frac{\varepsilon}{R} = \frac{\pi a^2}{R} \mu_0 N k$$

The solenoidal magnetic field points from left to right. An increase in the strength of the magnetic field will induce a current in the loop directed such that the magnetic field it produces point from right to left (Lenz's law). Therefore, the current flows from left to right through the resistor.

b) The change in the magnetic flux enclosed by the wire loop is equal to

$$\Delta\Phi = 2\pi a^2 \mu_0 N I$$

The current flowing through the resistor is equal to

$$I = \frac{\varepsilon}{R} = \frac{1}{R} \frac{d\Phi}{dt} = \frac{dQ}{dt}$$

This relation shows that

$$\Delta Q = \int_{-\infty}^{\infty} \frac{dQ}{dt} dt = -\frac{1}{R} \int_{-\infty}^{\infty} \frac{d\Phi}{dt} dt = -\frac{\Delta\Phi}{R}$$

Substituting the expression for $\Delta\Phi$ we obtain

$$|\Delta Q| = \frac{1}{R} (2\pi a^2 \mu_0 N I)$$

7.3. Inductance

Consider two loops: loop 1 and loop 2 (see Figure 7.6). A current I_1 flowing through loop 1 will produce a magnetic field at the position of loop 2 equal to

$$\vec{B}_1 = \frac{\mu_0}{4\pi} I_1 \oint \frac{d\vec{l}_1 \times \Delta\hat{r}}{\Delta r^2}$$

The magnetic flux through loop 2 is equal to

$$\Phi_2 = \int \vec{B}_1 \cdot d\vec{a}_2 = M_{21} I_1$$

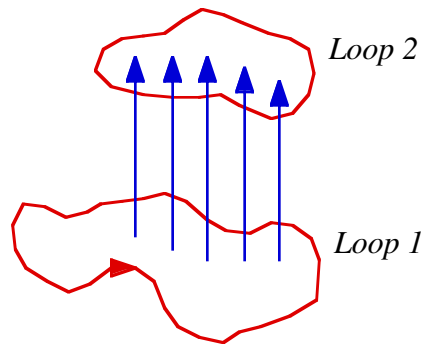


Figure 7.6. Inductance.

Here, M_{21} is called the **mutual inductance** of the two loops. It is a purely geometrical quantity that depends on the sizes, shapes and relative positions of the two loops. It does not change if we switch the role of loop 1 and loop 2: the flux through loop 2 when we run a current I around loop 1 is exactly the same as the flux through loop 1 when we send the same current I around loop 2.

Besides inducing an emf in a nearby loop, the changing current in loop 1 also induces an emf in loop 1. The flux through loop 1 generated by the current in loop 1 is equal to

$$\Phi_1 = L I_1$$

The constant of proportionality is called the **self inductance**. The unit of inductance is the **Henrie** (H).

Example: Problem 7.19

A square loop of wire, of side s , lies midway between two long wires, $3s$ apart and in the same place. (Actually, the long wires are sides of a large rectangular loop, but the short ends are so far away that they can be neglected.) A clockwise current I in the square loop is gradually increasing: $dI/dt = k = \text{constant}$. Find the emf induced in the big loop. Which way will the induced current flow?

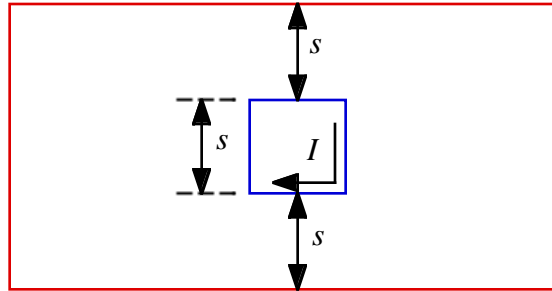


Figure 7.7. Problem 7.19.

The system is schematically shown in Figure 7.7. To calculate the emf induced in the large loop we need to determine the mutual inductance M . It is hard to calculate M in terms of a current in the square loop since the magnetic field generated by this loop is rather complicated (and therefore difficult to integrate). However, exploiting the equality of the mutual inductances, we can also evaluate the inductance in terms of a current I in the large loop. The magnetic field generated by the top wire of the large loop is that of an infinitely long straight wire carrying a current I is equal to

$$B = \frac{\mu_0 I}{2\pi r}$$

The flux associated with this magnetic field and intercepted by the square loop is equal to

$$\Phi_{top} = \frac{\mu_0}{2\pi} \int_s^{2s} \frac{I}{r} s dr = \frac{\mu_0}{2\pi} I s \ln 2$$

The magnetic field generated by the bottom wire at the position of the square loop will be pointing in the same direction as the magnetic field generated by the top wire at the position of the square loop. Since the square loop is located at the same distance from the top wire as it is from the bottom wire, the flux intercepted by the square loop is equal to

$$\Phi = 2\Phi_{top} = \frac{\mu_0}{\pi} I s \ln 2$$

Therefore, the mutual inductance of the two loops is equal to

$$M = \frac{\Phi}{I} = \frac{\mu_0}{\pi} s \ln 2$$

The induced emf in the large loop can now be calculated easily:

$$\varepsilon = -\frac{d\Phi}{dt} = -M \frac{dI}{dt} = -Mk = -\frac{\mu_0}{\pi} ks \ln 2$$

The direction of the flux through the large loop is pointing into the page. This can be seen most easily by considering the magnetic field lines. Inside the square loop, the field lines point into the page (right-hand rule). Since the field lines form closed loops, they must be pointing out of the page anywhere outside the square loop. However, the large wire loop only covers a limited fraction of space, and therefore definitely will not intercept all field lines outside the square loop. Therefore, there will be more field lines pointing into the page than there are field lines pointing out of the page. Consequently, the net magnetic flux will be pointing into the page. When the current in the square loop increases the flux intercepted by the large loop will increase. The induced emf will produce a magnetic field that counteracts this increase in flux, and therefore produces a flux pointing out of the paper. The right-hand rule shows that the direction of the current induced in the large loop must be flowing in a counter-clockwise direction.

7.4. The Maxwell Equations

The electric and magnetic fields in electrostatics and magnetostatics are described by the following four equations:

$$\bar{\nabla} \cdot \bar{E} = \frac{1}{\varepsilon_0} \rho \quad \text{Gauss' Law}$$

$$\bar{\nabla} \cdot \bar{B} = 0$$

$$\bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \quad \text{Faraday's Law}$$

$$\bar{\nabla} \times \bar{B} = \mu_0 \bar{J} \quad \text{Ampere's Law}$$

In systems with non-steady currents not all of these equations are valid anymore. For example,

$$\nabla \cdot (\nabla \times \vec{B}) = 0$$

for every vector function. However, according to Ampere's law

$$\nabla \cdot (\nabla \times \vec{B}) = \mu_0 (\nabla \cdot \vec{J})$$

which is only zero for steady currents (for which \vec{J} is a constant, independent of position). For non-steady currents

$$\mu_0 (\nabla \cdot \vec{J}) \neq 0 = \nabla \cdot (\nabla \times \vec{B})$$

We thus conclude that Ampere's law does not hold for non-steady currents. The failure of Ampere's law can also be observed in a system in which a capacitor is being charged (see Figure 7.8). During the charging process a current I is flowing through the wire, and consequently there will be a magnetic field present. The magnetic field generated by the charging current can be calculated using Ampere's law. When we are far away from the capacitor the generated magnetic field will be that of a line current. Consider an Amperian loop of radius r , centered on the wire. The line integral of \vec{B} around this loop is equal to

$$\oint \vec{B} \cdot d\vec{l} = 2\pi rB$$

According to Ampere's law the line integral of \vec{B} around a closed loop is proportional to the current intercepted by a surface spanned by this loop. For the system shown in Figure 7.8 the intercepted current is ill defined. Consider first surface 1. The current intercepted by surface 1 is equal to I . Surface 2 is also spanned by the Amperian loop, but the current intercepted by this loop is zero. We thus conclude that Ampere's law does not apply in systems where the current is not continuous.

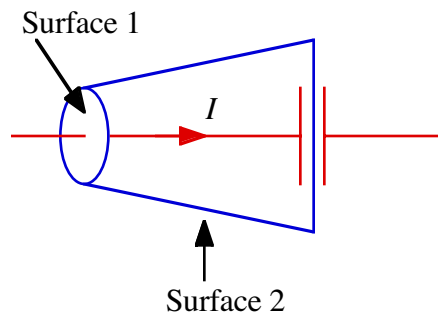


Figure 7.8. Charging a capacitor.

Maxwell modified Ampere's law in the following manner:

$$\nabla \times \vec{B} = \mu_0 \left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

The term added by Maxwell is called the **displacement current**. It is defined as

$$\vec{J}_d = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Consider the region between the capacitor plates in Figure 7.8. The electric field in this region is equal to

$$\vec{E} = \frac{\sigma}{\epsilon_0} \hat{k} = \frac{Q}{\epsilon_0 A} \hat{k}$$

where we have assumed that the field produced is that of an ideal capacitor with surface area A and the z axis is in the direction of the current. The rate of change of the electric field is equal to

$$\frac{\partial \vec{E}}{\partial t} = \frac{1}{\epsilon_0} \frac{\partial \sigma}{\partial t} \hat{k} = \frac{1}{\epsilon_0 A} \frac{\partial Q}{\partial t} \hat{k} = \frac{I}{\epsilon_0 A} \hat{k}$$

The surface integral of $\partial \vec{E} / \partial t$ across surface 2 is therefore equal to

$$\oint \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a} = \frac{I}{\epsilon_0}$$

The surface integral of $\nabla \times \vec{B}$ across surface 2 is equal to

$$\oint_{\text{Surface 2}} (\nabla \times \vec{B}) \cdot d\vec{a} = \mu_0 \oint_{\text{Surface 2}} \vec{J} \cdot d\vec{a} + \mu_0 \epsilon_0 \oint_{\text{Surface 2}} \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a} = \mu_0 I$$

The modification of Ampere's law by Maxwell insures that the surface integral of $\nabla \times \vec{B}$ is independent of the surface chosen. In electrostatics and magnetostatics the electric and magnetic fields are constant in time, and therefore, the new form of Ampere's law reduces to the form of Ampere's law we have been using so far.

In a region where there are no free charges or free currents Maxwell's equations become very symmetric

$$\begin{aligned}\bar{\nabla} \cdot \bar{E} &= 0 & \bar{\nabla} \cdot \bar{B} &= 0 \\ \bar{\nabla} \times \bar{E} &= -\frac{\partial \bar{B}}{\partial t} & \bar{\nabla} \times \bar{B} &= \mu_0 \epsilon_0 \frac{\partial \bar{E}}{\partial t}\end{aligned}$$

The symmetry is broken when electric charges are present, unless besides electric charges there are **magnetic monopoles**. If the magnetic charge density is equal to η and the magnetic current is equal to \bar{K} then Maxwell's equation become

$$\begin{aligned}\bar{\nabla} \cdot \bar{E} &= \frac{\rho}{\epsilon_0} & \bar{\nabla} \cdot \bar{B} &= \mu_0 \eta \\ \bar{\nabla} \times \bar{E} &= -\mu_0 \bar{K} - \frac{\partial \bar{B}}{\partial t} & \bar{\nabla} \times \bar{B} &= \mu_0 \bar{J} + \mu_0 \epsilon_0 \frac{\partial \bar{E}}{\partial t}\end{aligned}$$

To obtain Maxwell's equations that describe the electric and magnetic fields in matter we must take the bound charges and bound currents into account:

$$\begin{aligned}\rho_b &= -\bar{\nabla} \cdot \bar{P} \\ \bar{J}_b &= \bar{\nabla} \times \bar{M}\end{aligned}$$

In the non-static case, the polarization can be time dependent. Therefore, also the bound charge density is time dependent, and a net current can be associated with the change in the bound charge density. This current is called the **polarization current** \bar{J}_p and is equal to

$$\bar{J}_p = \frac{\partial \bar{P}}{\partial t}$$

Maxwell's equations in matter are therefore equal to

$$\bar{\nabla} \cdot \bar{E} = \frac{1}{\epsilon_0} (\rho_f + \rho_b) = \frac{1}{\epsilon_0} (\rho_f - \bar{\nabla} \cdot \bar{P}) \quad \text{Gauss' Law}$$

$$\bar{\nabla} \cdot \bar{B} = 0$$

$$\bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \quad \text{Faraday's Law}$$

$$\bar{\nabla} \times \bar{B} = \mu_0 (\bar{J}_f + \bar{J}_b + \bar{J}_p) + \mu_0 \epsilon_0 \frac{\partial \bar{E}}{\partial t} = \mu_0 \left(\bar{J}_f + \bar{\nabla} \times \bar{M} + \frac{\partial \bar{P}}{\partial t} \right) + \mu_0 \epsilon_0 \frac{\partial \bar{E}}{\partial t} \quad \text{Ampere's Law}$$

It is common to rewrite Maxwell's equations in terms of the parameters we can control (the free charge density and the free current density). Gauss's law can be rewritten as

$$\nabla \cdot (\epsilon_0 \bar{E} + \bar{P}) = \nabla \cdot \bar{D} = \rho_f$$

where \bar{D} is called the **electric displacement**. Ampere's law can be rewritten as

$$\nabla \times \left(\frac{\bar{B}}{\mu_0} - \bar{M} \right) = \nabla \times \bar{H} = \bar{J}_f + \frac{\partial}{\partial t} (\epsilon_0 \bar{E} + \bar{P}) = \bar{J}_f + \frac{\partial \bar{D}}{\partial t}$$

where \bar{H} is called the **H field**. The most general form of Maxwell's equations, in terms of the free charges and free currents, is given by

$$\begin{aligned} \nabla \cdot \bar{D} &= \rho_f & \nabla \cdot \bar{B} &= 0 \\ \nabla \times \bar{E} &= -\frac{\partial \bar{B}}{\partial t} & \nabla \times \bar{H} &= \bar{J}_f + \frac{\partial \bar{D}}{\partial t} \end{aligned}$$