Chapter 2. Electrostatics

2.1. The Electrostatic Field

To calculate the force exerted by some electric charges, q_1, q_2, q_3, ... (**the source charges**) on another charge Q (**the test charge**) we can use the **principle of superposition**. This principle states that the interaction between any two charges is completely unaffected by the presence of other charges. The force exerted on Q by q_1, q_2, and q_3 (see Figure 2.1) is therefore equal to the vector sum of the force $\overline{F}_1$ exerted by q_1 on Q, the force $\overline{F}_2$ exerted by q_2 on Q, and the force $\overline{F}_3$ exerted by q_3 on Q.

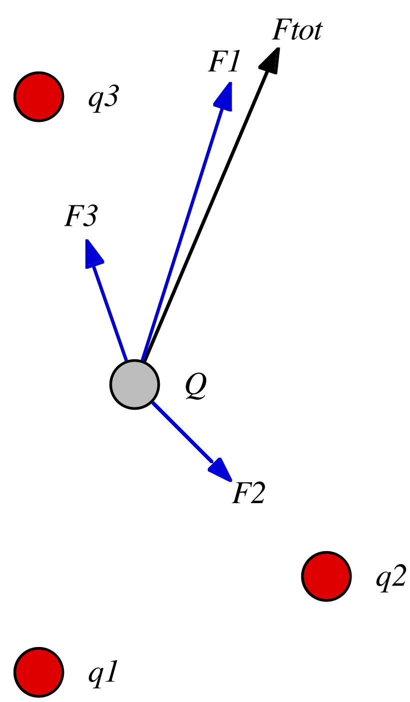

Figure 2.1. Superposition of forces.

The force exerted by a charged particle on another charged particle depends on their separation distance, on their velocities and on their accelerations. In this Chapter we will consider the special case in which the source charges are stationary.

The **electric field** produced by stationary source charges is called and **electrostatic field**. The electric field at a particular point is a vector whose magnitude is proportional to the total force acting on a test charge located at that point, and whose direction is equal to the direction of the force acting on a positive test charge. The electric field $\overline{E}$, generated by a collection of source charges, is defined as

$$\overline{E} = \frac{\overline{F}}{Q}$$

where $\overline{F}$ is the total electric force exerted by the source charges on the test charge Q. It is assumed that the test charge Q is small and therefore does not change the distribution of the source charges. The total force exerted by the source charges on the test charge is equal to

$$\overline{F} = \overline{F}_1 + \overline{F}_2 + \overline{F}_3 + ... = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 Q}{r_1^2}\hat{r}_1 + \frac{q_2 Q}{r_2^2}\hat{r}_2 + \frac{q_3 Q}{r_3^2}\hat{r}_3 + ...\right) = \frac{Q}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^2}\hat{r}_i$$

The electric field generated by the source charges is thus equal to

$$\overline{E} = \frac{\overline{F}}{Q} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^2}\hat{r}_i$$

In most applications the source charges are not discrete, but are distributed continuously over some region. The following three different distributions will be used in this course:

1. **line charge** λ: the charge per unit length.

2. **surface charge** σ: the charge per unit area.

3. **volume charge** ρ: the charge per unit volume.

To calculate the electric field at a point $\overline{P}$ generated by these charge distributions we have to replace the summation over the discrete charges with an integration over the continuous charge distribution:

1. for a line charge: $\overline{E}(\overline{P}) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int_{Line}\frac{\hat{r}}{r^2}\lambda dl$

2. for a surface charge: $\overline{E}(\overline{P}) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int_{Surface}\frac{\hat{r}}{r^2}\sigma da$

3. for a volume charge: $\overline{E}(\overline{P}) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int_{Volume}\frac{\hat{r}}{r^2}\rho d\tau$

Here $\hat{r}$ is the unit vector from a segment of the charge distribution to the point $\overline{P}$ at which we are evaluating the electric field, and r is the distance between this segment and point $\overline{P}$.

Example: Problem 2.2

a) Find the electric field (magnitude and direction) a distance z above the midpoint between two equal charges q a distance d apart. Check that your result is consistent with what you would expect when $z \gg d$.

b) Repeat part a), only this time make he right-hand charge $-q$ instead of $+q$.

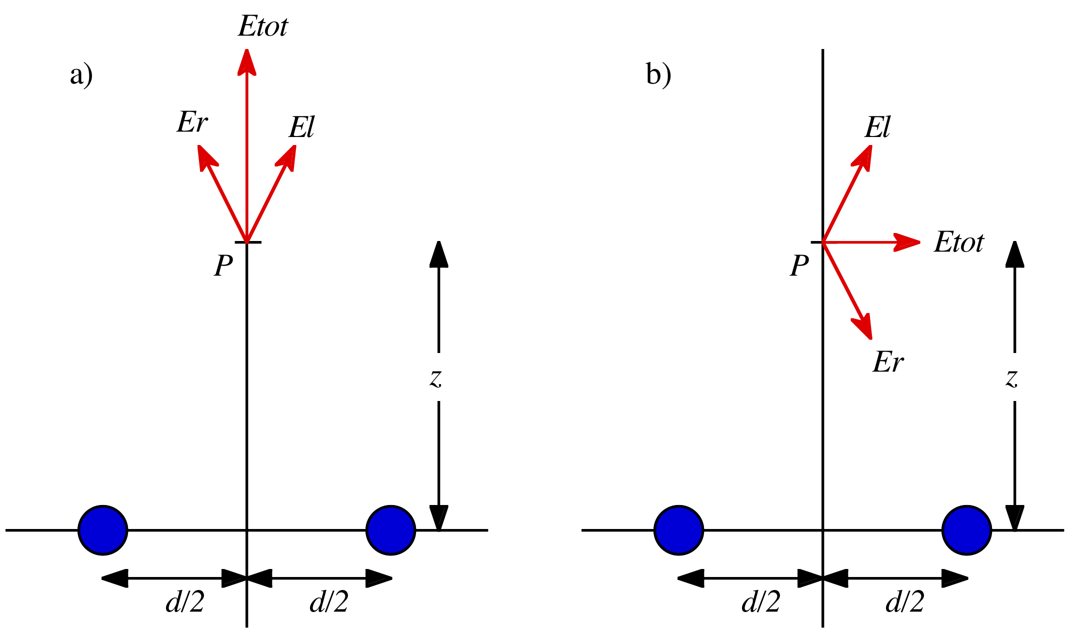

Figure 2.2. Problem 2.2

a) Figure 2.2a shows that the x components of the electric fields generated by the two point charges cancel. The total electric field at P is equal to the sum of the z components of the electric fields generated by the two point charges:

$$\overline{E}(\overline{P}) = 2\frac{1}{4\pi\varepsilon_0}\frac{q}{\left(\frac{1}{4}d^2 + z^2\right)}\frac{z}{\sqrt{\frac{1}{4}d^2 + z^2}}\hat{z} = \frac{1}{2\pi\varepsilon_0}\frac{qz}{\left(\frac{1}{4}d^2 + z^2\right)^{3/2}}\hat{z}$$

When $z \gg d$ this equation becomes approximately equal to

$$\overline{E}(\overline{P}) \cong \frac{1}{2\pi\varepsilon_0}\frac{q}{z^2}\hat{z} = \frac{1}{4\pi\varepsilon_0}\frac{2q}{z^2}\hat{z}$$

which is the Coulomb field generated by a point charge with charge $2q$.

b) For the electric fields generated by the point charges of the charge distribution shown in Figure 2.2b the z components cancel. The net electric field is therefore equal to

$$\overline{E}(\overline{P}) = 2\frac{1}{4\pi\varepsilon_0}\frac{q}{\left(\frac{1}{4}d^2 + z^2\right)}\frac{\frac{d}{2}}{\sqrt{\frac{1}{4}d^2 + z^2}}\hat{x} = \frac{1}{4\pi\varepsilon_0}\frac{qd}{\left(\frac{1}{4}d^2 + z^2\right)^{3/2}}\hat{x}$$

Example: Problem 2.5

Find the electric field a distance z above the center of a circular loop of radius r which carries a uniform line charge λ.

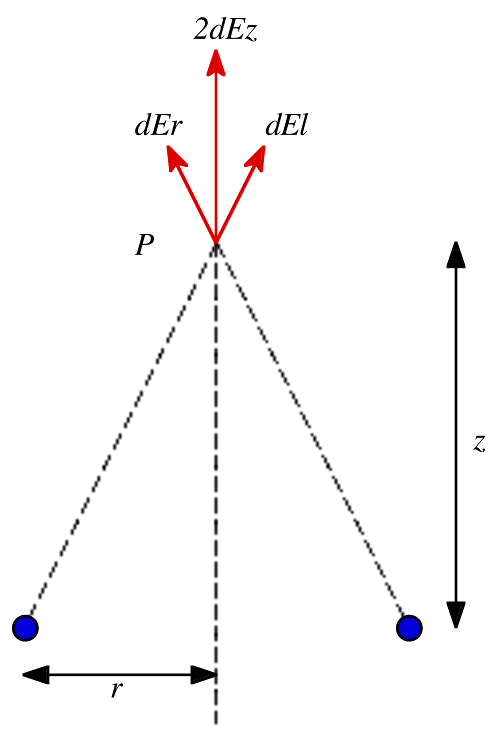

Figure 2.3. Problem 2.5.

Each segment of the loop is located at the same distance from P (see Figure 2.3). The magnitude of the electric field at P due to a segment of the ring of length dl is equal to

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dl}{r^2 + z^2}$$

When we integrate over the whole ring, the horizontal components of the electric field cancel. We therefore only need to consider the vertical component of the electric field generated by each segment:

$$dE_z = \frac{z}{\sqrt{r^2 + z^2}} dE = \frac{\lambda dl}{4\pi\varepsilon_0} \frac{z}{\left(r^2 + z^2\right)^{3/2}}$$

The total electric field generated by the ring can be obtained by integrating dE_z over the whole ring:

$$E = \frac{\lambda}{4\pi\varepsilon_0} \frac{z}{\left(r^2 + z^2\right)^{3/2}} \int_{Ring} dl = \frac{1}{4\pi\varepsilon_0} \frac{z}{\left(r^2 + z^2\right)^{3/2}} (2\pi r)\lambda = \frac{1}{4\pi\varepsilon_0} \frac{z}{\left(r^2 + z^2\right)^{3/2}} q$$

Example: Problem 2.7

Find the electric field a distance z from the center of a spherical surface of radius R, which carries a uniform surface charge density σ. Treat the case $z < R$ (inside) as well as $z > R$ (outside). Express your answer in terms of the total charge q on the surface.

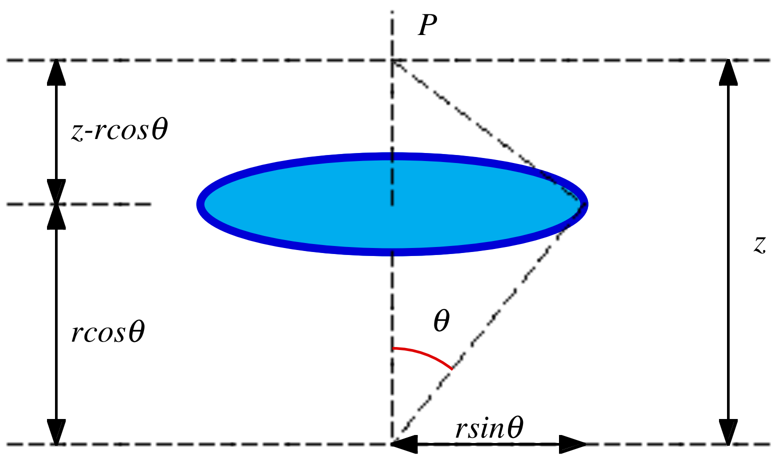

Figure 2.4. Problem 2.7.

Consider a slice of the shell centered on the z axis (see Figure 2.4). The polar angle of this slice is θ and its width is $d\theta$. The area dA of this ring is

$$dA = (2\pi r \sin\theta) r d\theta = 2\pi r^2 \sin\theta d\theta$$

The total charge on this ring is equal to

$$dq = \sigma dA = \frac{1}{2} q \sin\theta d\theta$$

where q is the total charge on the shell. The electric field produced by this ring at P can be calculated using the solution of Problem 2.5:

$$dE = \frac{1}{8\pi\varepsilon_0} \frac{q}{r} \frac{z - r\cos\theta}{\left(r^2 + z^2 - 2zr\cos\theta\right)^{3/2}} r\sin\theta d\theta$$

The total field at P can be found by integrating dE with respect to θ:

$$E = \frac{1}{8\pi\varepsilon_0} \frac{q}{r} \int_0^\pi \frac{z - r\cos\theta}{\left(r^2 + z^2 - 2zr\cos\theta\right)^{3/2}} r\sin\theta d\theta =$$

$$= \frac{1}{8\pi\varepsilon_0} \frac{q}{r} \int_0^\pi \frac{z - r\cos\theta}{\left(r^2 + z^2 - 2zr\cos\theta\right)^{3/2}} d(r\cos\theta) = \frac{1}{8\pi\varepsilon_0} \frac{q}{r} \int_{-r}^{r} \frac{z - y}{\left(r^2 + z^2 - 2zy\right)^{3/2}} dy$$

This integral can be solved using the following relation:

$$\frac{z - y}{\left(r^2 + z^2 - 2zy\right)^{3/2}} = -\frac{d}{dz} \frac{1}{\sqrt{r^2 + z^2 - 2zy}}$$

Substituting this expression into the integral we obtain:

$$E = -\frac{1}{8\pi\varepsilon_0} \frac{q}{r} \frac{d}{dz} \int_{-r}^{r} \frac{1}{\sqrt{r^2 + z^2 - 2zy}} dy = -\frac{1}{8\pi\varepsilon_0} \frac{q}{r} \frac{d}{dz} \left. \frac{\sqrt{r^2 + z^2 - 2zy}}{-z} \right|_{-r}^{r} =$$

$$= -\frac{1}{8\pi\varepsilon_0} \frac{q}{r} \frac{d}{dz} \left\{ \frac{(r + z) - |r - z|}{z} \right\}$$

Outside the shell, $z > r$ and consequently the electric field is equal to

$$E = -\frac{1}{8\pi\varepsilon_0} \frac{q}{r} \frac{d}{dz} \frac{(r + z) - (z - r)}{z} = -\frac{1}{4\pi\varepsilon_0} q \frac{d}{dz} \frac{1}{z} = \frac{1}{4\pi\varepsilon_0} \frac{q}{z^2}$$

Inside the shell, $z < r$ and consequently the electric field is equal to

$$E = -\frac{1}{8\pi\varepsilon_0} \frac{q}{r} \frac{d}{dz} \frac{(r + z) - (r - z)}{z} = -\frac{1}{4\pi\varepsilon_0} \frac{q}{r} \frac{d}{dz} 1 = 0$$

Thus the electric field of a charged shell is zero inside the shell. The electric field outside the shell is equal to the electric field of a point charge located at the center of the shell.

2.2. Divergence and Curl of Electrostatic Fields

The electric field can be graphically represented using field lines. The direction of the field lines indicates the direction in which a positive test charge moves when placed in this field. The density of field lines per unit area is proportional to the strength of the electric field. Field lines originate on positive charges and terminate on negative charges. Field lines can never cross since if this would occur, the direction of the electric field at that particular point would be undefined. Examples of field lines produced by positive point charges are shown in Figure 2.5.

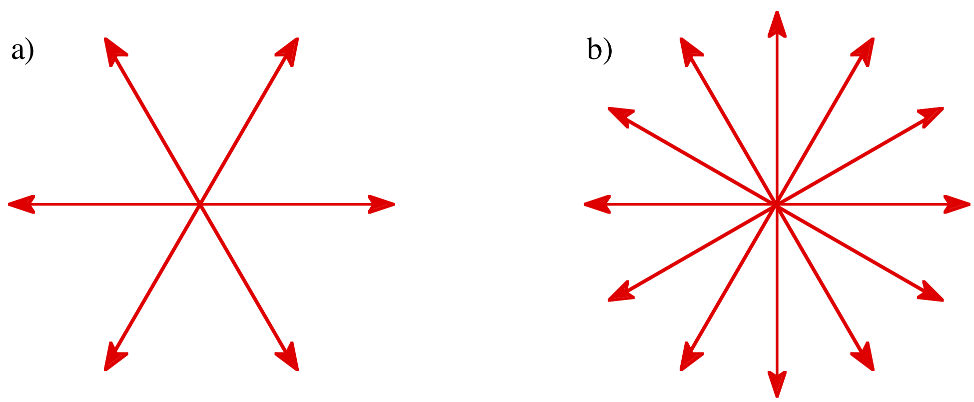

Figure 2.5. a) Electric field lines generated by a positive point charge with charge *q*. b) Electric field lines generated by a positive point charge with charge 2*q*.

The flux of electric field lines through any surface is proportional to the number of field lines passing through that surface. Consider for example a point charge q located at the origin. The electric flux Φ_E through a sphere of radius r, centered on the origin, is equal to

$$\Phi_E = \oint_{Surface} \overline{E} \bullet d\overline{a} = \frac{1}{4\pi\varepsilon_0} \oint_{Surface} \left(\frac{q}{r^2}\hat{r}\right) \bullet \left(r^2 \sin\theta d\theta d\phi \hat{r}\right) = \frac{q}{\varepsilon_0}$$

Since the number of field lines generated by the charge q depends only on the magnitude of the charge, any arbitrarily shaped surface that encloses q will intercept the same number of field lines. Therefore the electric flux through any surface that encloses the charge q is equal to q / ε_0. Using the principle of superposition we can extend our conclusion easily to systems containing more than one point charge:

$$\Phi_E = \oint_{Surface} \bar{E} \bullet d\bar{a} = \sum_i \oint_{Surface} \bar{E}_i \bullet d\bar{a} = \frac{1}{\varepsilon_0} \sum_i q_i$$

We thus conclude that for an arbitrary surface and arbitrary charge distribution

$$\oint_{Surface} \bar{E} \bullet d\bar{a} = \frac{Q_{enclosed}}{\varepsilon_0}$$

where Q_{enclosed} is the total charge enclosed by the surface. This is called **Gauss's law**. Since this equation involves an integral it is also called **Gauss's law in integral form**.

Using the divergence theorem the electric flux Φ_E can be rewritten as

$$\Phi_E = \oint_{Surface} \bar{E} \bullet d\bar{a} = \int_{Volume} \left(\bar{\nabla} \bullet \bar{E} \right) d\tau$$

We can also rewrite the enclosed charge Q_{encl} in terms of the charge density ρ:

$$Q_{enclosed} = \int_{Volume} \rho d\tau$$

Gauss's law can thus be rewritten as

$$\int_{Volume} \left(\bar{\nabla} \bullet \bar{E} \right) d\tau = \frac{1}{\varepsilon_0} \int_{Volume} \rho d\tau$$

Since we have not made any assumptions about the integration volume this equation must hold for any volume. This requires that the integrands are equal:

$$\bar{\nabla} \bullet \bar{E} = \frac{\rho}{\varepsilon_0}$$

This equation is called **Gauss's law in differential form**.

Gauss's law in differential form can also be obtained directly from Coulomb's law for a charge distribution $\rho(\bar{r}')$:

$$\bar{E}(\bar{r}') = \frac{1}{4\pi\varepsilon_0} \int_{Volume} \frac{\Delta\hat{r}}{(\Delta r)^2} \rho(\bar{r}') d\tau'$$

where $\Delta\bar{r} = \bar{r} - \bar{r}'$. The divergence of $\bar{E}(\bar{r})$ is equal to

$$\overline{\nabla} \bullet \overline{E}(\overline{r}) = \frac{1}{4\pi\varepsilon_0} \int_{Volume} \left(\overline{\nabla} \bullet \frac{\Delta\hat{r}}{(\Delta r)^2} \right) \rho(\overline{r}\,')d\tau' = \frac{1}{4\pi\varepsilon_0} \int_{Volume} 4\pi\delta^3(\overline{r} - \overline{r}\,')\rho(\overline{r}')d\tau' = \frac{\rho(\overline{r})}{\varepsilon_0}$$

which is Gauss's law in differential form. Gauss's law in integral form can be obtained by integrating $\overline{\nabla} \bullet \overline{E}(\overline{r})$ over the volume V:

$$\int_{Volume} \left(\overline{\nabla} \bullet \overline{E}(\overline{r})\right) d\tau = \int_{Surface} \overline{E} \bullet d\overline{a} = \Phi_E = \int_{Volume} \frac{\rho(\overline{r})}{\varepsilon_0} d\tau = \frac{Q_{Enclosed}}{\varepsilon_0}$$

Example: Problem 2.42

If the electric field in some region is given (in spherical coordinates) by the expression

$$\overline{E}(\overline{r}) = \frac{A\hat{r} + B\sin\theta\cos\phi\hat{\phi}}{r}$$

where A and B are constants, what is the charge density ρ?

The charge density ρ can be obtained from the given electric field, using Gauss's law in differential form:

$$\begin{aligned} \rho &= \varepsilon_0\left(\overline{\nabla} \bullet \overline{E}\right) = \varepsilon_0\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta E_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(E_\phi\right)\right) = \\ &= \varepsilon_0\left(\frac{1}{r^2}\frac{\partial}{\partial r}(Ar) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{B\sin\theta\cos\phi}{r}\right)\right) = \varepsilon_0\frac{A}{r^2} - \varepsilon_0\frac{B}{r^2}\sin\phi \end{aligned}$$

2.2.1. The curl of *E*

Consider a charge distribution $\rho(r)$. The electric field at a point P generated by this charge distribution is equal to

$$\overline{E}(\overline{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\Delta\hat{r}}{(\Delta r)^2}\rho(\overline{r}\,')d\tau'$$

where $\Delta\overline{r} = \overline{r} - \overline{r}\,'$. The curl of $\overline{E}$ is equal to

$$\overline{\nabla} \times \overline{E}(\overline{r}) = \frac{1}{4\pi\varepsilon_0}\int \left(\overline{\nabla} \times \frac{\Delta\hat{r}}{(\Delta r)^2}\right)\rho(\overline{r}\,')d\tau'$$

However, $\overline{\nabla} \times \hat{r} / r^2 = 0$ for every vector $\bar{r}$ and we thus conclude that

$$\overline{\nabla} \times \overline{E}(\bar{r}) = 0$$

2.2.2. Applications of Gauss's law

Although Gauss's law is always true it is only a useful tool to calculate the electric field if the charge distribution is symmetric:

1. If the charge distribution has **spherical symmetry**, then Gauss's law can be used with concentric spheres as Gaussian surfaces.

2. If the charge distribution has **cylindrical symmetry**, then Gauss's law can be used with coaxial cylinders as Gaussian surfaces.

3. If the charge distribution has **plane symmetr**y, then Gauss's law can be used with pill boxes as Gaussian surfaces.

Example: Problem 2.12

Use Gauss's law to find the electric field inside a uniformly charged sphere (charge density ρ) of radius R.

The charge distribution has spherical symmetry and consequently the Gaussian surface used to obtain the electric field will be a concentric sphere of radius r. The electric flux through this surface is equal to

$$\Phi_E = \oint_{Surface} \overline{E} \bullet d\bar{a} = 4\pi r^2 E(r)$$

The charge enclosed by this Gaussian surface is equal to

$$Q_{Enclosed} = \frac{4}{3}\pi r^3 \rho$$

Applying Gauss's law we obtain for the electric field:

$$E(r) = \frac{1}{4\pi r^2}\frac{Q_{Enclosed}}{\varepsilon_0} = \frac{1}{4\pi r^2}\frac{\frac{4}{3}\pi r^3 \rho}{\varepsilon_0} = \frac{\rho}{3\varepsilon_0} r$$

Example: Problem 2.14

Find the electric field inside a sphere which carries a charge density proportional to the distance from the origin: $\rho = k\, r$, for some constant k.

The charge distribution has spherical symmetry and we will therefore use a concentric sphere of radius r as a Gaussian surface. Since the electric field depends only on the distance r, it is constant on the Gaussian surface. The electric flux through this surface is therefore equal to

$$\Phi_E = \oint_{Surface} \bar{E} \bullet d\bar{a} = 4\pi r^2 E(r)$$

The charge enclosed by the Gaussian surface can be obtained by integrating the charge distribution between $r' = 0$ and $r' = r$:

$$Q_{Enclosed} = \int_{Volume} \rho(\bar{r}\,')d\tau = \int_0^r kr'\left(4\pi r'^2\right)dr' = \pi k r^4$$

Applying Gauss's law we obtain:

$$\Phi_E = 4\pi r^2 E(r) = \frac{Q_{Enclosed}}{\varepsilon_0} = \frac{\pi k r^4}{\varepsilon_0}$$

or

$$E(r) = \frac{\left(\dfrac{\pi k r^4}{\varepsilon_0}\right)}{4\pi r^2} = \frac{1}{4\varepsilon_0} k r^2$$

Example: Problem 2.16

A long coaxial cable carries a uniform (positive) volume charge density ρ on the inner cylinder (radius a), and uniform surface charge density on the outer cylindrical shell (radius b). The surface charge is negative and of just the right magnitude so that the cable as a whole is neutral. Find the electric field in each of the three regions: (1) inside the inner cylinder ($r < a$), (2) between the cylinders ($a < r < b$), (3) outside the cable ($b < r$).

The charge distribution has cylindrical symmetry and to apply Gauss's law we will use a cylindrical Gaussian surface. Consider a cylinder of radius r and length L. The electric field generated by the cylindrical charge distribution will be radially directed. As a consequence, there will be no electric flux going through the end caps of the cylinder (since here $\bar{E} \perp d\bar{a}$). The total electric flux through the cylinder is equal to

$$\Phi_E = \oint_{Surface} \bar{E} \bullet d\bar{a} = 2\pi r L E(r)$$

The enclosed charge must be calculated separately for each of the three regions:

1. $r < a$: $\quad Q_{Enclosed} = \pi r^2 L \rho$
2. $a < r < b$: $\quad Q_{Enclosed} = \pi a^2 L \rho$
3. $b < r$: $\quad Q_{Enclosed} = 0$

Applying Gauss's law we find

$$E(r) = \frac{1}{2\pi r L} \frac{Q_{Enclosed}}{\varepsilon_0}$$

Substituting the calculated Q_{encl} for the three regions we obtain

1. $r < a$: $\quad E(r) = \frac{1}{2\pi r L} \frac{Q_{Enclosed}}{\varepsilon_0} = \frac{1}{2\pi r L} \frac{\pi r^2 L \rho}{\varepsilon_0} = \frac{1}{2\varepsilon_0} r \rho$.
2. $a < r < b$: $\quad E(r) = \frac{1}{2\pi r L} \frac{Q_{Enclosed}}{\varepsilon_0} = \frac{1}{2\pi r L} \frac{\pi a^2 L \rho}{\varepsilon_0} = \frac{1}{2\varepsilon_0} \frac{a^2}{r} \rho$
3. $b < r$ $\quad E(r) = \frac{1}{2\pi r L} \frac{Q_{Enclosed}}{\varepsilon_0} = 0$

Example: Problem 2.18

Two spheres, each of radius R and carrying uniform charge densities of $+\rho$ and $-\rho$, respectively, are placed so that they partially overlap (see Figure 2.6). Call the vector from the negative center to the positive center $\bar{s}$. Show that the field in the region of overlap is constant and find its value.

To calculate the total field generated by this charge distribution we use the principle of superposition. The electric field generated by each sphere can be obtained using Gauss' law (see

Problem 2.12). Consider an arbitrary point in the overlap region of the two spheres (see Figure 2.7). The distance between this point and the center of the negatively charged sphere is r_-. The distance between this point and the center of the positively charged sphere is r_+. Figure 2.7 shows that the vector sum of $\bar{s}$ and $\bar{r}_+$ is equal to $\bar{r}_-$. Therefore,

$$\bar{r}_+ - \bar{r}_- = -\bar{s}$$

The total electric field at this point in the overlap region is the vector sum of the field due to the positively charged sphere and the field due to the negatively charged sphere:

$$\bar{E}_{tot} = \frac{\rho}{3\varepsilon_0}\left(\bar{r}_+ - \bar{r}_-\right)$$

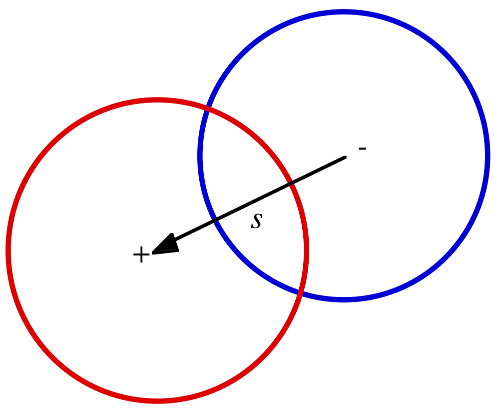

Figure 2.6. Problem 2.18.

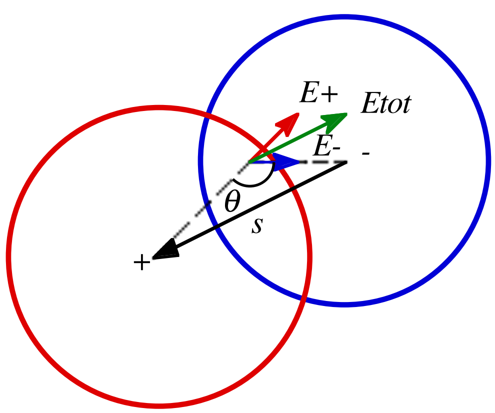

Figure 2.7. Calculation of E_{tot}.

The minus sign in front of $\bar{r}_-$ shows that the electric field generated by the negatively charged sphere is directed opposite to $\bar{r}_-$. Using the relation between $\bar{r}_+$ and $\bar{r}_-$ obtained from Figure 2.7 we can rewrite $\bar{E}_{tot}$ as

$$\bar{E}_{tot} = -\frac{\rho}{3\varepsilon_0}\bar{s}$$

which shows that the field in the overlap region is homogeneous and pointing in a direction opposite to $\bar{s}$.

2.3. The Electric Potential

The requirement that the curl of the electric field is equal to zero limits the number of vector functions that can describe the electric field. In addition, a theorem discussed in Chapter 1 states that any vector function whose curl is equal to zero is the gradient of a scalar function. The scalar function whose gradient is the electric field is called the **electric potential *V*** and it is defined as

$$\bar{E} = -\bar{\nabla} V$$

Taking the line integral of $\bar{\nabla} V$ between point *a* and point *b* we obtain

$$\int_a^b \bar{\nabla} V \bullet d\bar{l} = V(b) - V(a) = -\int_a^b \bar{E} \bullet d\bar{l}$$

Taking *a* to be the reference point and defining the potential to be zero there, we obtain for $V(b)$

$$V(b) = -\int_a^b \bar{E} \bullet d\bar{l}$$

The choice of the reference point *a* of the potential is arbitrary. Changing the reference point of the potential amounts to adding a constant to the potential:

$$V'(b) = -\int_{a'}^b \bar{E} \bullet d\bar{l} = -\int_{a'}^a \bar{E} \bullet d\bar{l} - \int_a^b \bar{E} \bullet d\bar{l} = K + V(b)$$

where *K* is a constant, independent of *b*, and equal to

$$K = -\int_{a'}^a \bar{E} \bullet d\bar{l}$$

However, since the gradient of a constant is equal to zero

$$E' = -\overline{\nabla} V' = -\overline{\nabla} V = E$$

Thus, the electric field generated by V' is equal to the electric field generated by V. The physical behavior of a system will depend only on the difference in electric potential and is therefore independent of the choice of the reference point. The most common choice of the reference point in electrostatic problems is infinity and the corresponding value of the potential is usually taken to be equal to zero:

$$V(b) = -\int_{\infty}^{b} \overline{E} \bullet d\bar{l}$$

The unit of the electrical potential is the Volt (V, 1V = 1 Nm/C).

Example: Problem 2.20

One of these is an impossible electrostatic field. Which one?

a) $E = k\left[(xy)\hat{i} + (2yz)\hat{j} + (3xz)\hat{k}\right]$

b) $E = k\left[(y^2)\hat{i} + (2xy + z^2)\hat{j} + (2yz)\hat{k}\right]$

Here, k is a constant with the appropriate units. For the *possible* one, find the potential, using the origin as your reference point. Check your answer by computing $\overline{\nabla} V$.

a) The curl of this vector function is equal to

$$\begin{aligned}\overline{\nabla} \times \overline{E} &= k\left(\frac{\partial}{\partial y}(3xz) - \frac{\partial}{\partial z}(2yz)\right)\hat{i} + k\left(\frac{\partial}{\partial z}(xy) - \frac{\partial}{\partial x}(3xz)\right)\hat{j} + \\ &\quad k\left(\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial y}(xy)\right) = k\left(-2y\hat{i} - 3z\hat{j} - x\hat{k}\right)\end{aligned}$$

Since the curl of this vector function is not equal to zero, this vector function can not describe an electric field.

b) The curl of this vector function is equal to

$$\begin{aligned}\overline{\nabla} \times \overline{E} &= k\left(\frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}(2xy + z^2)\right)\hat{i} + k\left(\frac{\partial}{\partial z}(y^2) - \frac{\partial}{\partial x}(2yz)\right)\hat{j} + \\ &\quad k\left(\frac{\partial}{\partial x}(2xy + z^2) - \frac{\partial}{\partial y}(y^2)\right) = 0\end{aligned}$$

Since the curl of this vector function is equal to zero it can describe an electric field. To calculate the electric potential V at an arbitrary point (x, y, z), using $(0, 0, 0)$ as a reference point, we have to evaluate the line integral of $\overline{E}$ between $(0, 0, 0)$ and (x, y, z). Since the line integral of $\overline{E}$ is path independent we are free to choose the most convenient integration path. I will use the following integration path:

$$(0,0,0) \rightarrow (x,0,0) \rightarrow (x,y,0) \rightarrow (x,y,z)$$

The first segment of the integration path is along the x axis:

$$d\bar{l} = dx\hat{i}$$

and

$$\overline{E} \bullet d\bar{l} = ky^2dx = 0$$

since $y = 0$ along this path. Consequently, the line integral of $\overline{E}$ along this segment of the integration path is equal to zero. The second segment of the path is parallel to the y axis:

$$d\bar{l} = dy\hat{j}$$

and

$$\overline{E} \bullet d\bar{l} = k\left(2xy + z^2\right)dy = 2kxydy$$

since $z = 0$ along this path. The line integral of $\overline{E}$ along this segment of the integration path is equal to

$$\int_{(x,0,0)}^{(x,y,0)} \overline{E} \bullet d\bar{l} = \int_0^y 2kxydy = kxy^2$$

The third segment of the integration path is parallel to the z axis:

$$d\bar{l} = dz\hat{k}$$

and

$$\overline{E} \bullet d\bar{l} = 2k\left(yz\right)dz$$

The line integral of $\overline{E}$ along this segment of the integration path is equal to

$$\int_{(x,y,0)}^{(x,y,z)} \overline{E} \bullet d\bar{l} = \int_0^z 2k(yz)dz = kyz^2$$

The electric potential at (x, y, z) is thus equal to

$$\begin{aligned} V(x,y,z) &= -\int_{(0,0,0)}^{(x,0,0)} \overline{E} \bullet d\bar{l} - \int_{(x,0,0)}^{(x,y,0)} \overline{E} \bullet d\bar{l} - \int_{(x,y,0)}^{(x,y,z)} \overline{E} \bullet d\bar{l} = \\ &= 0 - kxy^2 - kyz^2 = -k\left(xy^2 + yz^2\right) \end{aligned}$$

The answer can be verified by calculating the gradient of V:

$$\overline{\nabla} V = \frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k} = -k\left(y^2\hat{i} + \left(2xy + z^2\right)\hat{j} + \left(2yz\right)\hat{k}\right) = -\overline{E}$$

which is the opposite of the original electric field $\overline{E}$.

The advantage of using the electric potential V instead of the electric field is that V is a scalar function. The total electric potential generated by a charge distribution can be found using the superposition principle. This property follows immediately from the definition of V and the fact that the electric field satisfies the principle of superposition. Since

$$\overline{E} = \overline{E}_1 + \overline{E}_2 + \overline{E}_3 + ...$$

it follows that

$$V = -\int_\infty^b \overline{E} \bullet d\bar{l} = -\int_\infty^b \overline{E}_1 \bullet d\bar{l} - \int_\infty^b \overline{E}_2 \bullet d\bar{l} - \int_\infty^b \overline{E}_3 \bullet d\bar{l} - ... = V_1 + V_2 + V_3 + ...$$

This equation shows that the total potential at any point is the algebraic sum of the potentials at that point due to all the source charges separately. This ordinary sum of scalars is in general easier to evaluate then a vector sum.

Example: Problem 2.46

Suppose the electric potential is given by the expression

$$V(\bar{r}) = A\frac{e^{-\lambda r}}{r}$$

for all r (A and λ are constants). Find the electric field $\overline{E}(\bar{r})$, the charge density $\rho(\bar{r})$, and the total charge Q.

The electric field $\overline{E}(\overline{r})$ can be immediately obtained from the electric potential:

$$\overline{E}(\overline{r}) = -\overline{\nabla} V(\overline{r}) = -\frac{\partial}{\partial r}\left(A\frac{e^{-\lambda r}}{r}\right)\hat{r} = \left(\lambda A\frac{e^{-\lambda r}}{r} + A\frac{e^{-\lambda r}}{r^2}\right)\hat{r}$$

The charge density $\rho(\overline{r})$ can be found using the electric field $\overline{E}(\overline{r})$ and the following relation:

$$\overline{\nabla} \bullet \overline{E}(\overline{r}) = \frac{\rho(\overline{r})}{\varepsilon_0}$$

This expression shows that

$$\rho(\overline{r}) = \varepsilon_0\left[\overline{\nabla} \bullet \overline{E}(\overline{r})\right]$$

Substituting the expression for the electric field $\overline{E}(\overline{r})$ we obtain for the charge density $\rho(\overline{r})$:

$$\begin{aligned}
\rho(\overline{r}) &= \varepsilon_0 A\left[\overline{\nabla} \bullet \left((1+\lambda r)e^{-\lambda r}\frac{\hat{r}}{r^2}\right)\right] = \\
&= \varepsilon_0 A\left[(1+\lambda r)e^{-\lambda r}\left(\overline{\nabla} \bullet \frac{\hat{r}}{r^2}\right) + \frac{\hat{r}}{r^2} \bullet \overline{\nabla}\left((1+\lambda r)e^{-\lambda r}\right)\right] = \\
&= \varepsilon_0 A\left[4\pi(1+\lambda r)e^{-\lambda r}\delta^3(\overline{r}) - \frac{\lambda^2 e^{-\lambda r}}{r}\right] = \varepsilon_0 A\left[4\pi\delta^3(\overline{r}) - \frac{\lambda^2 e^{-\lambda r}}{r}\right]
\end{aligned}$$

The total charge Q can be found by volume integration of $\rho(\overline{r})$:

$$\begin{aligned}
Q_{tot} = \int_{Volume} \rho(\overline{r})d\tau &= \int_0^\infty \varepsilon_0 A\left[4\pi\delta^3(\overline{r}) - \frac{\lambda^2 e^{-\lambda r}}{r}\right]4\pi r^2 dr = \\
&= 4\pi\varepsilon_0 A\left[\int_0^\infty 4\pi\delta^3(\overline{r})r^2 dr - \int_0^\infty r\lambda^2 e^{-\lambda r}dr\right] = \\
&= -4\pi\varepsilon_0 A\int_0^\infty r\lambda^2 e^{-\lambda r}dr
\end{aligned}$$

The integral can be solved easily:

$$\int_0^\infty re^{-\lambda r}dr = -\frac{d}{d\lambda}\int_0^\infty e^{-\lambda r}dr = -\frac{d}{d\lambda}\left(\frac{1}{\lambda}\right) = \frac{1}{\lambda^2}$$

The total charge is thus equal to

$$Q_{tot} = -4\pi\varepsilon_0 A$$

The charge distribution $\rho(\bar{r})$ can be directly used to obtained from the electric potential $V(\bar{r})$

$$\rho(\bar{r}) = \varepsilon_0\left[\bar{\nabla} \bullet \bar{E}(\bar{r})\right] = -\varepsilon_0\left[\bar{\nabla} \bullet \bar{\nabla} V(\bar{r})\right] = -\varepsilon_0 \bar{\nabla}^2 V(\bar{r})$$

This equation can be rewritten as

$$\bar{\nabla}^2 V(\bar{r}) = -\frac{\rho(\bar{r})}{\varepsilon_0}$$

and is known as **Poisson's equation**. In the regions where $\rho(\bar{r}) = 0$ this equation reduces to **Laplace's equation**:

$$\bar{\nabla}^2 V(\bar{r}) = 0$$

The electric potential generated by a discrete charge distribution can be obtained using the principle of superposition:

$$V_{tot}(\bar{r}) = \sum_{i=1}^{n} V_i(\bar{r})$$

where $V_i(\bar{r})$ is the electric potential generated by the point charge q_i. A point charge q_i located at the origin will generate an electric potential $V_i(\bar{r})$ equal to

$$V_i(\bar{r}) = -\frac{1}{4\pi\varepsilon_0}\int_{\infty}^{r} \frac{q_i}{r'^2}\, dr' = \frac{1}{4\pi\varepsilon_0}\frac{q_i}{r}$$

In general, point charge q_i will be located at position $\bar{r}_i$ and the electric potential generated by this point charge at position $\bar{r}$ is equal to

$$V_i(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q_i}{\left|\bar{r} - \bar{r}_i\right|}$$

The total electric potential generated by the whole set of point charges is equal to

$$V_{tot}(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{\left|\bar{r} - \bar{r}_i\right|}$$

To calculate the electric potential generated by a continuous charge distribution we have to replace the summation over point charges with an integration over the continuous charge distribution. For the three charge distributions we will be using in this course we obtain:

1. line charge λ: $V_{tot}(\bar{r}) = \dfrac{1}{4\pi\varepsilon_0} \displaystyle\int_{Line} \dfrac{\lambda}{|\bar{r} - \bar{r}'|} dl'$

2. surface charge σ: $V_{tot}(\bar{r}) = \dfrac{1}{4\pi\varepsilon_0} \displaystyle\int_{Surface} \dfrac{\sigma}{|\bar{r} - \bar{r}'|} da'$

3. volume charge ρ: $V_{tot}(\bar{r}) = \dfrac{1}{4\pi\varepsilon_0} \displaystyle\int_{Volume} \dfrac{\rho}{|\bar{r} - \bar{r}'|} d\tau'$

Example: Problem 2.25

Using the general expression for V in terms of ρ find the potential at a distance z above the center of the charge distributions of Figure 2.8. In each case, compute $\bar{E} = -\bar{\nabla} V$. Suppose that we changed the right-hand charge in Figure 2.8a to $-q$. What is then the potential at P? What field does this suggest? Compare your answer to Problem 2.2b, and explain carefully any discrepancy.

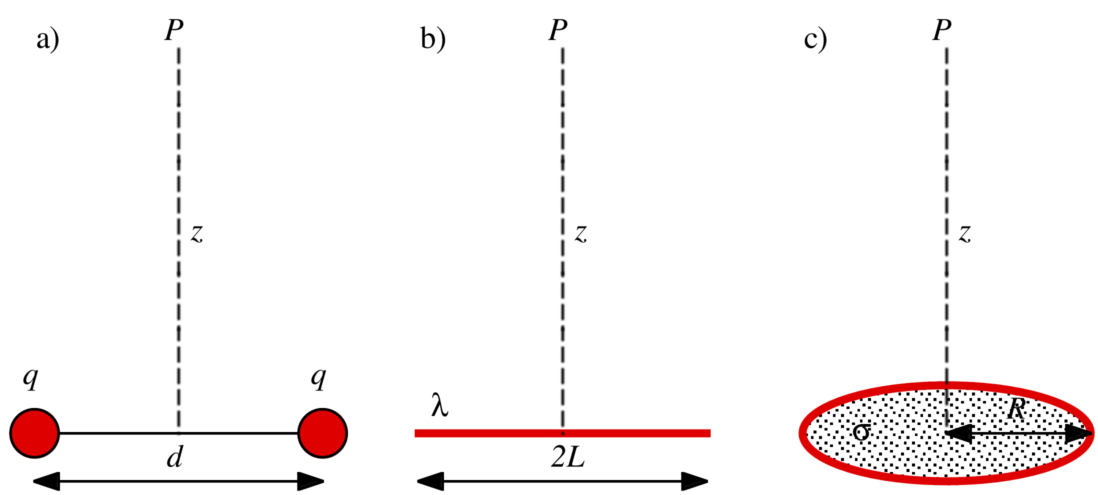

Figure 2.8. Problem 2.35.

a) The electric potential at P generated by the two point charges is equal to

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{\frac{1}{4}d^2 + z^2}} + \frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{\frac{1}{4}d^2 + z^2}} = \frac{1}{2\pi\varepsilon_0}\frac{q}{\sqrt{\frac{1}{4}d^2 + z^2}}$$

The electric field generated by the two point charges can be obtained by taking the gradient of the electric potential:

$$\overline{E} = -\overline{\nabla}V = -\frac{\partial}{\partial z}\left(\frac{1}{2\pi\varepsilon_0}\frac{q}{\sqrt{\frac{1}{4}d^2 + z^2}}\right)\hat{k} = \frac{1}{2\pi\varepsilon_0}\frac{qz}{\left(\frac{1}{4}d^2 + z^2\right)^{3/2}}\hat{k}$$

If we change the right-hand charge to -*q* then the total potential at *P* is equal to zero. However, this does not imply that the electric field at *P* is equal to zero. In our calculation we have assumed right from the start that $x = 0$ and $y = 0$. Obviously, the potential at *P* will therefore not show an *x* and *y* dependence. This however not necessarily indicates that the components of the electric field along the *x* and *y* direction are zero. This can be demonstrated by calculating the general expression for the electric potential of this charge distribution at an arbitrary point (x,y,z):

$$\begin{aligned} V(x,y,z) &= \frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{\left(x+\frac{1}{2}d\right)^2 + y^2 + z^2}} + \frac{1}{4\pi\varepsilon_0}\frac{-q}{\sqrt{\left(x-\frac{1}{2}d\right)^2 + y^2 + z^2}} = \\ &= \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{\left(x+\frac{1}{2}d\right)^2 + y^2 + z^2}} - \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{\left(x-\frac{1}{2}d\right)^2 + y^2 + z^2}} \end{aligned}$$

The various components of the electric field can be obtained by taking the gradient of this expression:

$$\overline{E}_x(x,y,z) = -\frac{\partial V}{\partial x} = \frac{q}{4\pi\varepsilon_0}\frac{\left(x+\frac{1}{2}d\right)}{\left(\left(x+\frac{1}{2}d\right)^2 + y^2 + z^2\right)^{3/2}} - \frac{q}{4\pi\varepsilon_0}\frac{\left(x-\frac{1}{2}d\right)}{\left(\left(x-\frac{1}{2}d\right)^2 + y^2 + z^2\right)^{3/2}}$$

$$\overline{E}_y(x,y,z) = -\frac{\partial V}{\partial y} = \frac{q}{4\pi\varepsilon_0}\frac{y}{\left(\left(x+\frac{1}{2}d\right)^2 + y^2 + z^2\right)^{3/2}} - \frac{q}{4\pi\varepsilon_0}\frac{y}{\left(\left(x-\frac{1}{2}d\right)^2 + y^2 + z^2\right)^{3/2}}$$

$$\overline{E}_z(x,y,z) = -\frac{\partial V}{\partial z} = \frac{q}{4\pi\varepsilon_0}\frac{z}{\left(\left(x+\frac{1}{2}d\right)^2 + y^2 + z^2\right)^{3/2}} - \frac{q}{4\pi\varepsilon_0}\frac{z}{\left(\left(x-\frac{1}{2}d\right)^2 + y^2 + z^2\right)^{3/2}}$$

The components of the electric field at $P = (0, 0, z)$ can now be calculated easily:

$$\overline{E}_x(0,0,z) = \frac{q}{4\pi\varepsilon_0}\frac{d}{\left(\frac{1}{4}d^2 + z^2\right)^{3/2}}$$

$$\overline{E}_y(0,0,z) = 0$$

$$\overline{E}_z(0,0,z) = 0$$

b) Consider a small segment of the rod, centered at position x and with length dx. The charge on this segment is equal to λdx. The potential generated by this segment at P is equal to

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dx}{\sqrt{x^2 + z^2}}$$

The total potential generated by the rod at P can be obtained by integrating dV between $x = -L$ and $x = L$

$$V = \int_{-L}^{L}\frac{1}{4\pi\varepsilon_0}\frac{\lambda dx}{\sqrt{x^2 + z^2}} = \frac{\lambda}{4\pi\varepsilon_0}\left[\ln\left(\sqrt{L^2 + z^2} + L\right) - \ln\left(\sqrt{L^2 + z^2} - L\right)\right]$$

The z component of the electric field at P can be obtained from the potential V by calculating the z component of the gradient of V. We obtain

$$\begin{aligned} E_z(x,y,z) &= -\frac{\lambda}{4\pi\varepsilon_0}\frac{\partial}{\partial z}\left[\ln\left(\sqrt{L^2 + z^2} + L\right) - \ln\left(\sqrt{L^2 + z^2} - L\right)\right] = \\ &= -\frac{\lambda}{4\pi\varepsilon_0}\left[\frac{\frac{z}{\sqrt{L^2 + z^2}}}{\sqrt{L^2 + z^2} + L} - \frac{\frac{z}{\sqrt{L^2 + z^2}}}{\sqrt{L^2 + z^2} - L}\right] = \frac{\lambda}{4\pi\varepsilon_0}\frac{2L}{z\sqrt{L^2 + z^2}} \end{aligned}$$

c) Consider a ring of radius r and width dr. The charge on this ring is equal to

$$dq = \sigma\left[\pi(r + dr)^2 - \pi r^2\right] = 2\pi \cdot \sigma \cdot r dr$$

The electric potential dV at P generated by this ring is equal to

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{\sqrt{r^2+z^2}} = \frac{\sigma}{2\varepsilon_0}\frac{rdr}{\sqrt{r^2+z^2}}$$

The total electric potential at P can be obtained by integrating dV between $r = 0$ and $r = R$:

$$V = \frac{\sigma}{2\varepsilon_0}\int_0^R \frac{rdr}{\sqrt{r^2+z^2}} = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{R^2+z^2} - z\right]$$

The z component of the electric field generated by this charge distribution can be obtained by taking the gradient of V:

$$E_z = -\frac{\sigma}{2\varepsilon_0}\frac{\partial}{\partial z}\left[\sqrt{R^2+z^2} - z\right] = -\frac{\sigma}{2\varepsilon_0}\left[\frac{z}{\sqrt{R^2+z^2}} - 1\right]$$

Example: Problem 2.5

Find the electric field a distance z above the center of a circular loop of radius r, which carries a uniform line charge λ.

The total charge Q on the ring is equal to

$$Q = 2\pi R\lambda$$

The total electric potential V at P is equal to

$$V = \frac{1}{4\pi\varepsilon_0}\frac{2\pi R\lambda}{\sqrt{R^2+z^2}} = \frac{\lambda}{2\varepsilon_0}\frac{R}{\sqrt{R^2+z^2}}$$

The z component of the electric field at P can be obtained by calculating the gradient of V:

$$E_z = -\frac{\lambda}{2\varepsilon_0}\frac{\partial}{\partial z}\frac{R}{\sqrt{R^2+z^2}} = \frac{\lambda}{2\varepsilon_0}\frac{Rz}{\left(R^2+z^2\right)^{3/2}}$$

This is the same answer we obtained in the beginning of this Chapter by taking the vector sum of the segments of the ring.

We have seen so far that there are three fundamental quantities of electrostatics:

1. The **charge density** ρ

2. The **electric field** $\bar{E}$

3. The **electric potential** V

If one of these quantities is known, the others can be calculated:

$Known \downarrow$	ρ	$\bar{E}$	V
ρ		$\bar{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\hat{r}}{r^2}\rho d\tau$	$V = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho}{r} d\tau$
$\bar{E}$	$\rho = \varepsilon_0 \bar{\nabla} \bullet \bar{E}$		$V = -\int \bar{E} \bullet d\bar{l}$
V	$\rho = -\varepsilon_0 \bar{\nabla}^2 V$	$\bar{E} = -\bar{\nabla} V$	

In general the charge density ρ and the electric field $\bar{E}$ do not have to be continuous. Consider for example an infinitesimal thin charge sheet with surface charge σ. The relation between the electric field above and below the sheet can be obtained using Gauss's law. Consider a rectangular box of height ε and area A (see Figure 2.9). The electric flux through the surface of the box, in the limit $\varepsilon \to 0$, is equal to

$$\Phi_E = \int_{Surface} \bar{E} \bullet d\bar{a} = \left(E_{\perp above} - E_{\perp below}\right)A$$

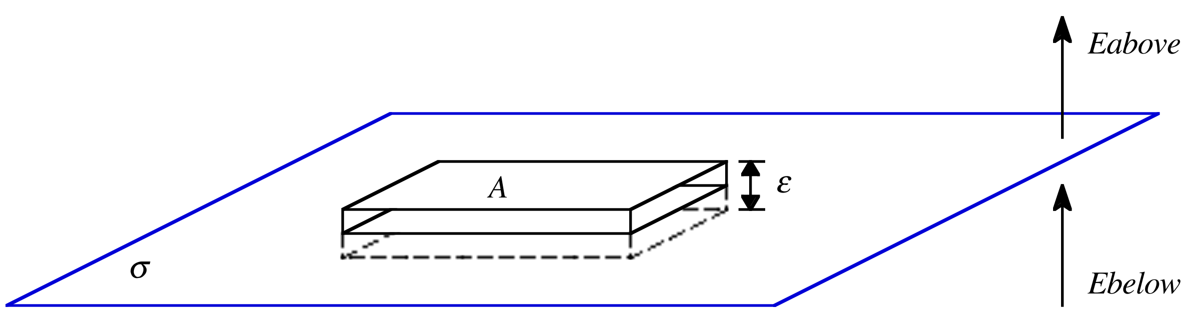

Figure 2.9. Electric field near a charge sheet.

where $E_{\perp,above}$ and $E_{\perp,below}$ are the perpendicular components of the electric field above and below the charge sheet. Using Gauss's law and the rectangular box shown in Figure 2.9 as integration volume we obtain

$$\frac{Q_{encl}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0} = \left(E_{\perp,above} - E_{\perp,below}\right)A$$

This equation shows that the electric field perpendicular to the charge sheet is discontinuous at the boundary. The difference between the perpendicular component of the electric field above and below the charge sheet is equal to

$$E_{\perp,above} - E_{\perp,below} = \frac{\sigma}{\varepsilon_0}$$

The tangential component of the electric field is always continuous at any boundary. This can be demonstrated by calculating the line integral of $\overline{E}$ around a rectangular loop of length L and height ε (see Figure 2.10). The line integral of $\overline{E}$, in the limit $\varepsilon \to 0$, is equal to

$$\oint \overline{E} \bullet d\bar{l} = \int \overline{E}_{\parallel,above} \bullet d\bar{l} + \int \overline{E}_{\parallel,below} \bullet d\bar{l} = \left(E_{\parallel,above} - E_{\parallel,below}\right)L$$

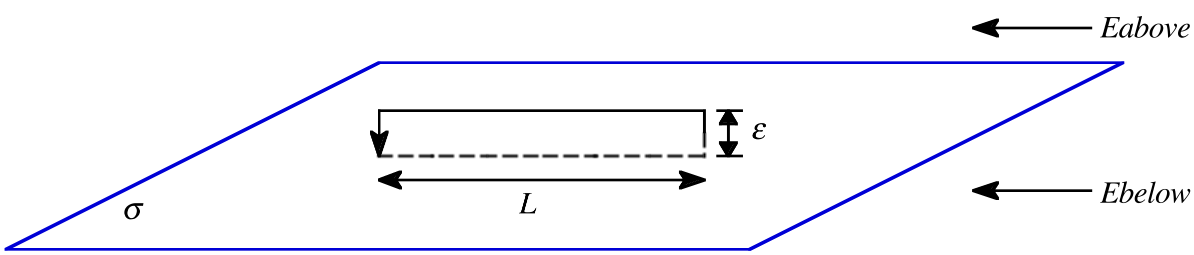

Figure 2.10. Parallel field close to charge sheet.

Since the line integral of $\overline{E}$ around any closed loop is zero we conclude that

$$\left(E_{\parallel,above} - E_{\parallel,below}\right)L = 0$$

or

$$E_{\parallel,above} = E_{\parallel,below}$$

These boundary conditions for $\overline{E}$ can be combined into a single formula:

$$\overline{E}_{above} - \overline{E}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

where $\hat{n}$ is a unit vector perpendicular to the surface and pointing towards the *above* region.

The electric potential is continuous across any boundary. This is a direct results of the definition of V in terms of the line integral of $\overline{E}$:

$$V_{above} - V_{below} = \int_{above}^{below} \overline{E} \bullet d\bar{l}$$

If the path shrinks the line integral will approach zero, independent of whether $\overline{E}$ is continuous or discontinuous. Thus

$$V_{above} = V_{below}$$

Example: Problem 2.30

a) Check that the results of examples 4 and 5 of Griffiths are consistent with the boundary conditions for $\overline{E}$.
b) Use Gauss's law to find the field inside and outside a long hollow cylindrical tube which carries a uniform surface charge σ. Check that your results are consistent with the boundary conditions for $\overline{E}$.
c) Check that the result of example 7 of Griffiths is consistent with the boundary conditions for V.

a) **<u>Example 4 (Griffiths)</u>**: The electric field generated by an infinite plane carrying a uniform surface charge σ is directed perpendicular to the sheet and has a magnitude equal to

$$\overline{E}_{above} = \frac{\sigma}{2\varepsilon_0}\hat{k}$$

$$\overline{E}_{below} = -\frac{\sigma}{2\varepsilon_0}\hat{k}$$

Therefore,

$$\overline{E}_{above} - \overline{E}_{below} = \frac{\sigma}{\varepsilon_0}\hat{k} = \frac{\sigma_{sheet}}{\varepsilon_0}\hat{k}$$

which is in agreement with the boundary conditions for $\overline{E}$.

Example 5 (Griffiths): The electric field generated by the two charge sheets is directed perpendicular to the sheets and has a magnitude equal to

$$\overline{E}_I = 0$$

$$\overline{E}_{II} = \frac{\sigma}{\varepsilon_0}\hat{i}$$

$$\overline{E}_{III} = 0$$

The change in the strength of the electric field at the left sheet is equal to

$$\overline{E}_{II} - \overline{E}_I = \frac{\sigma}{\varepsilon_0}\hat{i} = \frac{\sigma_{left}}{\varepsilon_0}\hat{i}$$

The change in the strength of the electric field at the right sheet is equal to

$$\overline{E}_{III} - \overline{E}_{II} = -\frac{\sigma}{\varepsilon_0}\hat{i} = \frac{\sigma_{right}}{\varepsilon_0}\hat{i}$$

These relations show agreement with the boundary conditions for $\overline{E}$.

b) Consider a Gaussian surface of length L and radius r. As a result of the symmetry of the system, the electric field will be directed radially. The electric flux through this Gaussian surface is therefore equal to the electric flux through its curved surface which is equal to

$$\Phi_E = \int \overline{E} \bullet d\overline{a} = 2\pi\, r\, L\, E(r)$$

The charge enclosed by the Gaussian surface is equal to zero when $r < R$. Therefore

$$E(\overline{r}) = 0$$

when $r < R$. When $r > R$ the charge enclosed by the Gaussian surface is equal to

$$Q_{enclosed} = 2\pi\, R\, L\, \sigma$$

The electric field for $r > R$, obtained using Gauss' law, is equal to

$$\overline{E}(\overline{r}) = \frac{1}{2\pi\, r\, L}\frac{Q_{enclosed}}{\varepsilon_0}\hat{r} = \frac{\sigma}{\varepsilon_0}\frac{R}{r}\hat{r}$$

The magnitude of the electric field just outside the cylinder, directed radially, is equal to

$$\overline{E}_{outside} = \overline{E}(\overline{R}) = \frac{\sigma}{\varepsilon_0}\hat{r}$$

The magnitude of the electric field just inside the cylinder is equal to

$$\overline{E}_{inside} = 0$$

Therefore,

$$\overline{E}_{outside} - \overline{E}_{inside} = \frac{\sigma}{\varepsilon_0}\hat{r}$$

which is consistent with the boundary conditions for E.

c) **<u>Example 7 (Griffiths)</u>**: the electric potential just outside the charged spherical shell is equal to

$$V_{outside} = V_{outside}(z = R) = \frac{R^2\sigma}{\varepsilon_0 R} = \frac{\sigma R}{\varepsilon_0}$$

The electric potential just inside the charged spherical shell is equal to

$$V_{inside} = V_{inside}(z = R) = \frac{\sigma R}{\varepsilon_0}$$

These two equations show that the electric potential is continuous at the boundary.

2.4. Work and Energy in Electrostatics

Consider a point charge q_1 located at the origin. A point charge q_2 is moved from infinity to a point a distance r_2 from the origin. We will assume that the point charge q_1 remains fixed at the origin when point charge q_2 is moved. The force exerted by q_1 on q_2 is equal to

$$\overline{F}_{12} = q_2\overline{E}_1$$

where $\overline{E}_1$ is the electric field generated by q_1. In order to move charge q_2 we will have to exert a force opposite to $\overline{F}_{12}$. Therefore, the total work that must be done to move q_2 from infinity to r_2 is equal to

$$W = -\int_{\infty}^{r_2} \bar{F}_{12} \bullet d\bar{l} = -\int_{\infty}^{r_2} q_2 \bar{E}_1 \bullet d\bar{l} = q_2 V_1(\bar{r}_2)$$

where $V_1(\bar{r}_2)$ is the electric potential generated by q_1 at position r_2. Using the equation of V for a point charge, the work W can be rewritten as

$$W = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r_2}$$

This work W is the work necessary to assemble the system of two point charges and is also called the **electrostatic potential energy** of the system. The energy of a system of more than two point charges can be found in a similar manner using the superposition principle. For example, for a system consisting of three point charges (see Figure 2.11) the electrostatic potential energy is equal to

$$W = \frac{1}{4\pi\varepsilon_0} \left\{ \frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right\}$$

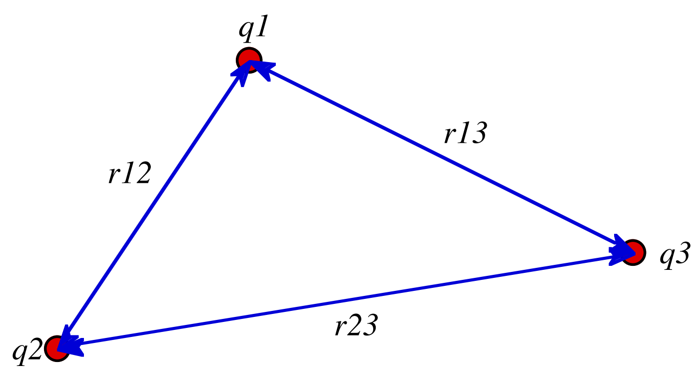

Figure 2.11. System of three point charges.

In this equation we have added the electrostatic energies of each pair of point charges. The general expression of the electrostatic potential energy for n point charges is

$$W = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{q_i q_j}{r_{ij}}$$

The lower limit of j (= i + 1) insures that each pair of point charges is only counted once. The electrostatic potential energy can also be written as

$$W = \frac{1}{2}\left[\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{q_i\, q_j}{r_{ij}}\right] = \frac{1}{2}\sum_{i=1}^{n} q_i \sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{4\pi\varepsilon_0}\frac{q_j}{r_{ij}} = \frac{1}{2}\sum_{i=1}^{n} q_i V_i$$

where V_i is the electrostatic potential at the location of q_i due to all other point charges.

When the charge of the system is not distributed as point charges, but rather as a continuous charge distribution ρ, then the electrostatic potential energy of the system must be rewritten as

$$W = \frac{1}{2}\int_{Volume} \rho V d\tau$$

For continuous surface and line charges the electrostatic potential energy is equal to

$$W = \frac{1}{2}\int_{Surface} \sigma V da$$

and

$$W = \frac{1}{2}\int_{Line} \lambda V dl$$

However, we have already seen in this Chapter that ρ, V, and $\overline{E}$ carry the same equivalent information. The charge density ρ, for example, is related to the electric field $\overline{E}$:

$$\rho = \varepsilon_0\left(\overline{\nabla}\bullet\overline{E}\right)$$

Using this relation we can rewrite the electrostatic potential energy as

$$\begin{aligned} W &= \frac{\varepsilon_0}{2}\int_{Volume}\left(\overline{\nabla}\bullet\overline{E}\right)V d\tau = \frac{\varepsilon_0}{2}\int_{Volume}\left(\overline{\nabla}\bullet\left(V\overline{E}\right)\right)d\tau - \frac{\varepsilon_0}{2}\int_{Volume}\left(\overline{E}\bullet\left(\overline{\nabla}V\right)\right)d\tau = \\ &= \frac{\varepsilon_0}{2}\int_{Surface} V\overline{E}\bullet d\overline{a} + \frac{\varepsilon_0}{2}\int_{Volume}\left(\overline{E}\bullet\overline{E}\right)d\tau \end{aligned}$$

where we have used one of the product rules of vector derivatives and the definition of $\overline{E}$ in terms of V. In deriving this expression we have not made any assumptions about the volume considered. This expression is therefore valid for any volume. If we consider all space, then the contribution of the surface integral approaches zero since $V\overline{E}$ will approach zero faster than $1/r^2$. Thus the total electrostatic potential energy of the system is equal to

$$W = \frac{\varepsilon_0}{2} \int_{Volume} E^2 d\tau$$

Example: Problem 2.45

A sphere of radius R carries a charge density $\rho(\bar{r}) = k\,r$ (where k is a constant). Find the energy of the configuration. Check your answer by calculating it in at least two different ways.

Method 1:

The first method we will use to calculate the electrostatic potential energy of the charged sphere uses the volume integral of E^2 to calculate W. The electric field generated by the charged sphere can be obtained using Gauss's law. We will use a concentric sphere of radius r as the Gaussian surface. First consider the case in which $r < R$. The charge enclosed by the Gaussian surface can be obtained by volume integration of the charge distribution:

$$Q_{Enclosed} = \int_{Sphere} \rho(\bar{r}) d\tau = 4\pi \int_0^r k\,r\,r^2 dr = \pi\,k\,r^4$$

The electric flux through the Gaussian surface is equal to

$$\Phi_E = 4\pi\,r^2 E(r)$$

Applying Gauss's law we find for the electric field inside the sphere ($r < R$):

$$E(r) = \frac{1}{4\pi\,r^2}\frac{Q_{Enclosed}}{\varepsilon_0} = \frac{1}{4\pi\,r^2}\frac{\pi\,k\,r^4}{\varepsilon_0} = \frac{k\,r^2}{4\varepsilon_0}$$

The electric field outside the sphere ($r > R$) can also be obtained using Gauss's law:

$$E(r) = \frac{1}{4\pi\,r^2}\frac{Q_{Enclosed}}{\varepsilon_0} = \frac{1}{4\pi\,r^2}\frac{\pi\,k\,R^4}{\varepsilon_0} = \frac{k\,R^4}{4\varepsilon_0\,r^2}$$

The total electrostatic energy can be obtained from the electric field:

$$W = \frac{1}{2}\varepsilon_0 \int_{All\ Space} E^2(r) d\tau = 2\pi\varepsilon_0 \int_0^R \left[\frac{k\,r^2}{4\varepsilon_0}\right]^2 r^2 dr + 2\pi\varepsilon_0 \int_0^R \left[\frac{k\,R^4}{4\varepsilon_0\,r^2}\right]^2 r^2 dr =$$

$$= \frac{\pi k^2}{8\varepsilon_0}\int_0^R r^6 dr + \frac{\pi k^2}{8\varepsilon_0}\int_R^\infty \frac{R^8}{r^2} dr = \frac{\pi k^2}{8\varepsilon_0}\left[\frac{1}{7}R^7 + R^7\right] = \frac{\pi k^2 R^7}{7\varepsilon_0}$$

Method 2:

An alternative way calculate the electrostatic potential energy is to use the following relation:

$$W = \frac{1}{2}\int_{Volume} \rho V d\tau$$

The electrostatic potential V can be obtained immediately from the electric field $\overline{E}$. To evaluate the volume integral of ρV we only need to know the electrostatic potential V inside the charged sphere:

$$V(\bar{r}) = -\int_{\infty}^{r} \overline{E} \bullet d\bar{l} = -\frac{1}{4\varepsilon_0}\left[\int_{\infty}^{R} \frac{k\,R^4}{r^2} dr + \int_{R}^{r} kr^2 dr\right] = \frac{k}{12\varepsilon_0}\left(4R^3 - r^3\right)$$

The electrostatic potential energy of the system is thus equal to

$$W = \frac{1}{2}\int_{Sphere} \rho(\bar{r})V(\bar{r})d\tau = \frac{4\pi}{2}\int_{0}^{R} k\,r \frac{k}{12\varepsilon_0}\left(4R^3 - r^3\right)r^2 dr = \frac{4\pi}{2}\frac{k^2}{12\varepsilon_0}\left(R^7 - \frac{1}{7}R^7\right) = \frac{\pi\,k^2}{7\varepsilon_0}R^7$$

which is equal to the energy calculated using method 1.

2.5. Metallic Conductors

In a **metallic conductor** one or more electrons per atom are free to move around through the material. Metallic conductors have the following electrostatic properties:

1. **The electric field inside the conductor is equal to zero.**
   If there would be an electric field inside the conductor, the free charges would move and produce an electric field of their own opposite to the initial electric field. Free charges will continue to flow until the cancellation of the initial field is complete.

2. **The charge density inside a conductor is equal to zero.**
   This property is a direct result of property 1. If the electric field inside a conductor is equal to zero, then the electric flux through any arbitrary closed surface inside the conductor is equal to zero. This immediately implies that the charge density inside the conductor is equal to zero everywhere (Gauss's law).

3. **Any net charge of a conductor resides on the surface.**

Since the charge density inside a conductor is equal to zero, any net charge can only reside on the surface.

4. **The electrostatic potential *V* is constant throughout the conductor.**
Consider two arbitrary points a and b inside a conductor (see Figure 2.12). The potential difference between a and b is equal to

$$V(b) - V(a) = -\int_a^b \overline{E} \bullet d\bar{l}$$

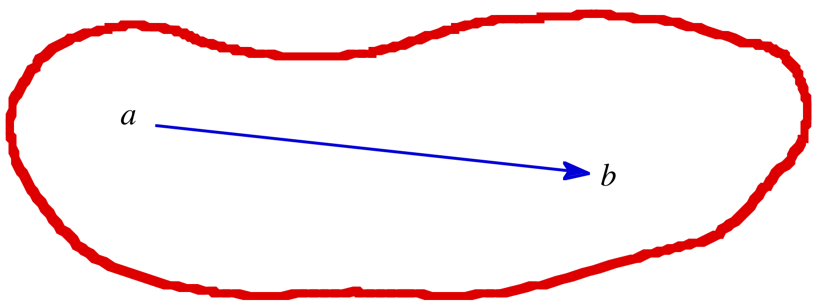

Figure 2.12. Potential inside metallic conductor.

Since the electric field inside a conductor is equal to zero, the line integral of $\overline{E}$ between a and b is equal to zero. Thus

$$V(b) - V(a) = 0$$

or

$$V(b) = V(a)$$

5. **The electric field is perpendicular to the surface, just outside the conductor.**
If there would be a tangential component of the electric field at the surface, then the surface charge would immediately flow around the surface until it cancels this tangential component.

Example: A spherical conducting shell

a) Suppose we place a point charge q at the center of a neutral spherical conducting shell (see Figure 2.13). It will attract negative charge to the inner surface of the conductor. How much induced charge will accumulate here?
b) Find E and V as function of r in the three regions $r < a$, $a < r < b$, and $r > b$.

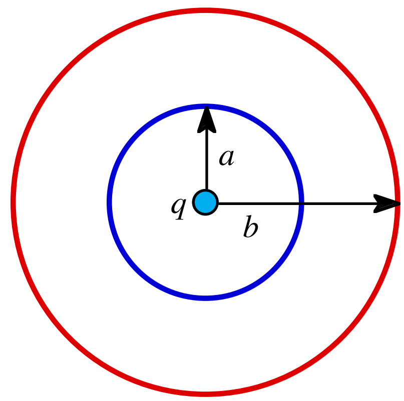

Figure 2.13. A spherical conducting shell.

a) The electric field inside the conducting shell is equal to zero (property 1 of conductors). Therefore, the electric flux through any concentric spherical Gaussian surface of radius r ($a<r<b$) is equal to zero. However, according to Gauss's law this implies that the charge enclosed by this surface is equal to zero. This can only be achieved if the charge accumulated on the inside of the conducting shell is equal to $-q$. Since the conducting shell is neutral and any net charge must reside on the surface, the charge on the outside of the conducting shell must be equal to $+q$.

b) The electric field generated by this system can be calculated using Gauss's law. In the three different regions the electric field is equal to

$$E(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \qquad \text{for } b < r$$

$$E(r) = 0 \qquad \text{for } a < r < b$$

$$E(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \qquad \text{for } r < a$$

The electrostatic potential $V(r)$ can be obtained by calculating the line integral of $\overline{E}$ from infinity to a point a distance r from the origin. Taking the reference point at infinity and setting the value of the electrostatic potential to zero there we can calculate the electrostatic potential. The line integral of $\overline{E}$ has to be evaluated for each of the three regions separately.

For $b < r$:

$$V(r) = -\int_\infty^r \overline{E}(r) \bullet d\bar{l} = -\frac{1}{4\pi\varepsilon_0}\int_\infty^r \frac{q}{r'^2}dr' = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

For $a < r < b$:

$$V(r) = -\int_{\infty}^{r} \overline{E}(r) \bullet d\bar{l} = -\frac{1}{4\pi\varepsilon_0}\int_{\infty}^{b} \frac{q}{r'^2} dr' = \frac{1}{4\pi\varepsilon_0}\frac{q}{b}$$

For $r < a$:

$$V(r) = -\int_{\infty}^{r} \overline{E}(r) \bullet d\bar{l} = -\frac{1}{4\pi\varepsilon_0}\int_{\infty}^{b} \frac{q}{r'^2} dr' - \frac{1}{4\pi\varepsilon_0}\int_{a}^{r} \frac{q}{r'^2} dr' = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{b} - \frac{q}{a} + \frac{q}{r}\right]$$

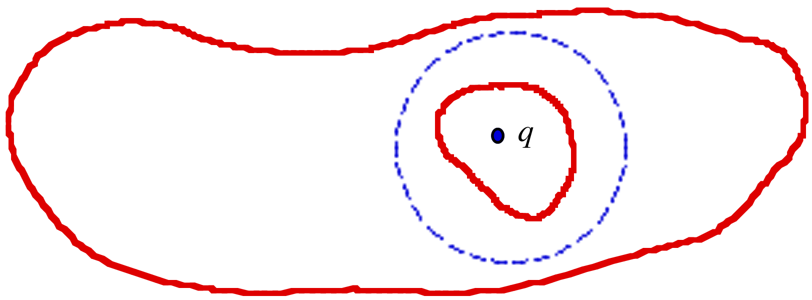

Figure 2.14. Arbitrarily shaped conductor.

In this example we have looked at a symmetric system but the general conclusions are also valid for an arbitrarily shaped conductor. For example, consider the conductor with a cavity shown in Figure 2.14. Consider also a Gaussian surface that completely surrounds the cavity (see for example the dashed line in Figure 2.14). Since the electric field inside the conductor is equal to zero, the electric flux through the Gaussian surface is equal to zero. Gauss's law immediately implies that the charge enclosed by the surface is equal to zero. Therefore, if there is a charge q inside the cavity there will be an induced charge equal to $-q$ on the walls of the cavity. On the other hand, if there is no charge inside the cavity then there will be no charge on the walls of the cavity. In this case, the electric field inside the cavity will be equal to zero. This can be demonstrated by assuming that the electric field inside the cavity is not equal to zero. In this case, there must be at least one field line inside the cavity. Since field lines originate on a positive charge and terminate on a negative charge, and since there is no charge inside the cavity, this field line must start and end on the cavity walls (see for example Figure 2.15). Now consider a closed loop, which follows the field line inside the cavity and has an arbitrary shape inside the conductor (see Figure 2.15). The line integral of $\overline{E}$ inside the cavity is definitely not equal to zero since the magnitude of $\overline{E}$ is not equal to zero and since the path is defined such that $\overline{E}$ and $d\bar{l}$ are parallel. Since the electric field inside the conductor is equal to zero, the path integral of $\overline{E}$ inside the conductor is equal to zero. Therefore, the path integral of $\overline{E}$ along the path indicated in Figure 2.15 is not equal to zero if the magnitude of $\overline{E}$ is not equal to zero inside the cavity. However, the line integral of $\overline{E}$ along any closed path must be equal to zero and consequently the electric field inside the cavity must be equal to zero.

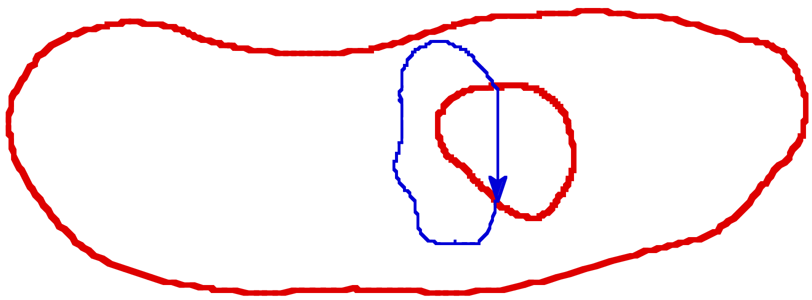

Figure 2.15. Field line in cavity.

Example: Problem 2.35

A metal sphere of radius R, carrying charge q, is surrounded by a thick concentric metal shell (inner radius a, outer radius b, see Figure 2.16). The shell carries no net charge.

a) Find the surface charge density σ at R, at a, and at b.

b) Find the potential at the center of the sphere, using infinity as reference.

c) Now the outer surface is touched to a grounding wire, which lowers its potential to zero (same as at infinity). How do your answers to a) and b) change?

a) Since the net charge of a conductor resides on its surface, the charge q of the metal sphere will reside its surface. The charge density on this surface will therefore be equal to

$$\sigma_R = \frac{q}{4\pi R^2}$$

As a result of the charge on the metal sphere there will be a charge equal to -q induced on the inner surface of the metal shell. Its surface charge density will therefore be equal to

$$\sigma_a = \frac{-q}{4\pi a^2}$$

Since the metal shell is neutral there will be a charge equal to +q on the outside of the shell. The surface charge density on the outside of the shell will therefore be equal to

$$\sigma_b = \frac{q}{4\pi b^2}$$

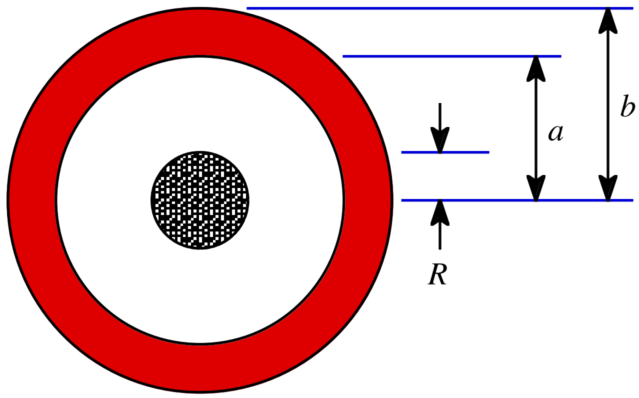

Figure 2.16. Problem 2.35.

b) The potential at the center of the metal sphere can be found by calculating the line integral of $\bar{E}$ between infinity and the center. The electric field in the regions outside the sphere and shell can be found using Gauss's law. The electric field inside the shell and sphere is equal to zero. Therefore,

$$V_{center} = -\int_{\infty}^{0} \bar{E} \bullet d\bar{l} = -\frac{q}{4\pi\varepsilon_0}\int_{\infty}^{b} \frac{1}{r^2}\,dr - \frac{q}{4\pi\varepsilon_0}\int_{a}^{R} \frac{1}{r^2}\,dr = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{b} - \frac{1}{a} + \frac{1}{R}\right]$$

c) When the outside of the shell is grounded, the charge density on the outside will become zero. The charge density on the inside of the shell and on the metal sphere will remain the same. The electric potential at the center of the system will also change as a result of grounding the outer shell. Since the electric potential of the outer shell is zero, we do not need to consider the line integral of $\bar{E}$ in the region outside the shell to determine the potential at the center of the sphere. Thus

$$V_{center} = -\int_{b}^{0} \bar{E} \bullet d\bar{l} = -\frac{q}{4\pi\varepsilon_0}\int_{a}^{R} \frac{1}{r^2}\,dr = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{R} - \frac{1}{a}\right]$$

Consider a conductor with surface charge σ, placed in an external electric field. Because the electric field inside the conductor is zero, the boundary conditions for the electric field require that the field immediately above the conductor is equal to

$$\overline{E}_{above} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

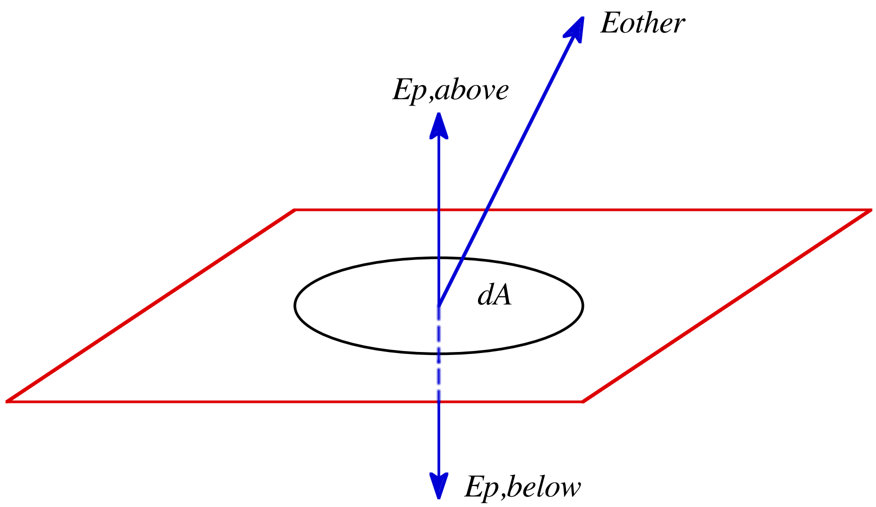

Figure 2.17. Patch of surface of conductor.

This electric field will exert a force on the surface charge.

Consider a small, infinitely thin, patch of the surface with surface area *dA* (see Figure 2.17). The electric field directly above and below the patch is equal to the vector sum of the electric field generated by the patch, the electric field generated by the rest of the conductor and the external electric field. The electric field generated by the patch is equal to

$$\overline{E}_{p,above} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

$$\overline{E}_{p,below} = -\frac{\sigma}{\varepsilon_0}\hat{n}$$

The remaining field, $\overline{E}_{other}$, is continuous across the patch, and consequently the total electric field above and below the patch is equal to

$$\overline{E}_{above} = \overline{E}_{other} + \frac{\sigma}{\varepsilon_0}\hat{n}$$

$$\overline{E}_{below} = \overline{E}_{other} - \frac{\sigma}{\varepsilon_0}\hat{n}$$

These two equations show that $\overline{E}_{other}$ is equal to

$$\overline{E}_{other} = \frac{1}{2}\left(\overline{E}_{above} + \overline{E}_{below}\right)$$

In this case the electric field below the surface is equal to zero and the electric field above the surface is directly determined by the boundary condition for the electric field at the surface. Thus

$$\overline{E}_{other} = \frac{1}{2}\left(\frac{\sigma}{\varepsilon_0}\hat{n} + 0\right) = \frac{\sigma}{2\varepsilon_0}\hat{n}$$

Since the patch cannot exert a force on itself, the electric force exerted on it is entirely due to the electric field $\overline{E}_{other}$. The charge on the patch is equal to $\sigma\, dA$. Therefore, the force exerted on the patch is equal to

$$d\overline{F} = \sigma\, dA\, \overline{E}_{other} = \frac{\sigma^2}{2\varepsilon_0} dA\, \hat{n}$$

The force per unit area of the conductor is equal to

$$f = \frac{d\overline{F}}{dA} = \frac{\sigma^2}{2\varepsilon_0}\hat{n}$$

This equation can be rewritten in terms of the electric field just outside the conductor as

$$f = \frac{1}{2\varepsilon_0}\left(\varepsilon_0 E\right)^2 \hat{n} = \frac{\varepsilon_0}{2} E^2 \hat{n}$$

This force is directed outwards. It is called the **radiation pressure**.

2.6. Capacitors

Consider two conductors (see Figure 2.18), one with a charge equal to +Q and one with a charge equal to -Q. The potential difference between the two conductors is equal to

$$\Delta V = V_+ - V_- = -\int_-^+ \overline{E} \bullet d\bar{l}$$

Since the electric field $\overline{E}$ is proportional to the charge Q, the potential difference ΔV will also be proportional to Q. The constant of proportionality is called the **capacitance** $\boldsymbol{C}$ of the system and is defined as

$$C = \frac{Q}{\Delta V}$$

Figure 2.18. Two conductors.

The capacitance C is determined by the size, the shape, and the separation distance of the two conductors. The unit of capacitance is the **farad** (***F***). The capacitance of a system of conductors can in general be calculated by carrying out the following steps:

1. Place a charge $+Q$ on one of the conductors. Place a charge of $-Q$ on the other conductor (for a two conductor system).
2. Calculate the electric field in the region between the two conductors.
3. Use the electric field calculated in step 2 to calculate the potential difference between the two conductors.
4. Apply the result of part 3 to calculate the capacitance:

$$C = \frac{|Q|}{|\Delta V|}$$

We will now discuss two examples in which we follow these steps to calculate the capacitance.

Example: Example 2.11 (Griffiths)
Find the capacitance of two concentric shells, with radii a and b.

Place a charge $+Q$ on the inner shell and a charge $-Q$ on the outer shell. The electric field between the shells can be found using Gauss's law and is equal to

$$\bar{E}(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r} \qquad a < r < b$$

The potential difference between the outer shell and the inner shell is equal to

$$V(a) - V(b) = -\int_a^b \bar{E}(\bar{r}) \bullet d\bar{l} = -\int_a^b \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}dr = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

The capacitance of this system is equal to

$$C = \frac{Q}{|\Delta V|} = \frac{Q}{\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)} = 4\pi\varepsilon_0\frac{ab}{b-a}$$

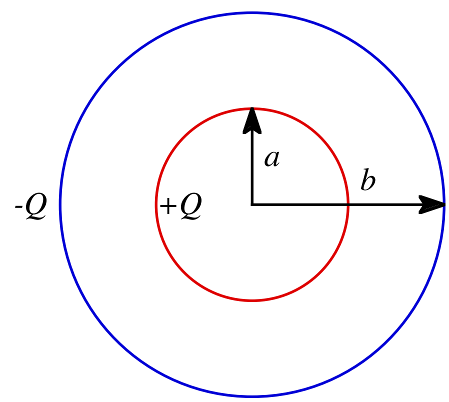

Figure 2.19. Example 2.11.

A system does not have to have two conductors in order to have a capacitance. Consider for example a single spherical shell of radius R. The capacitance of this system of conductors can be calculated by following the same steps as in Example 12. First of all, put a charge Q on the conductor. Gauss's law can be used to calculate the electric field generated by this system with the following result:

$$\bar{E}(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$$

Taking infinity as the reference point we can calculate the electrostatic potential on the surface of the shell:

$$V(R) = -\int_\infty^R \bar{E}(\bar{r}) \bullet d\bar{l} = -\int_\infty^R \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}dr = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R}$$

Therefore, the capacitance of the shell is equal to

$$C = \frac{Q}{|\Delta V|} = \frac{Q}{\dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{R}} = 4\pi\varepsilon_0 R$$

Let us now consider a parallel-plate capacitor. The work required to charge up the parallel-plate capacitor can be calculated in various ways:

<u>Method 1</u>: Since we are free to chose the reference point and reference value of the potential we will chose it such that the potential of the positively charges plate is $+\Delta V / 2$ and the potential of the negatively charged plate is $-\Delta V / 2$. The energy of this charge distribution is then equal to

$$W = \frac{1}{2}\int \rho V d\tau = \frac{1}{2}\left[Q\frac{\Delta V}{2} + \left(-Q\right)\left(\frac{-\Delta V}{2}\right)\right] = \frac{1}{2}Q(\Delta V) = \frac{1}{2}C(\Delta V)^2$$

<u>Method 2</u>: Let us look at the charging process in detail. Initially both conductors are uncharged and $\Delta V = 0$. At some intermediate step in the charging process the charge on the positively charged conductor will be equal to q. The potential difference between the conductors will then be equal to

$$\Delta V = \frac{q}{C}$$

To increase the charge on the positively charged conductor by dq we have to move this charge dq across this potential difference ΔV. The work required to do this is equal to

$$dW = \Delta V\, dq = \frac{q}{C} dq$$

Therefore, the total work required to charge up the capacitor from $q = 0$ to $q = Q$ is equal to

$$W = \int_0^Q \frac{q}{C} dq = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}C\left(\Delta V\right)^2$$

Example: Problem 2.40.

Suppose the plates of a parallel-plate capacitor move closer together by an infinitesimal distance ε, as a result of their mutual attraction.

a) Use equation (2.45) of Griffiths to express the amount of work done by electrostatic forces, in terms of the field E and the area of the plates A.
b) Use equation (2.40) of Griffiths to express the energy lost by the field in this process.

a) We will assume that the parallel-plate capacitor is an ideal capacitor with a homogeneous electric field E between the plates and no electric field outside the plates. The electrostatic force per unit surface area is equal to

$$\bar{f} = \frac{\varepsilon_0}{2} E^2 \hat{n}$$

The total force exerted on each plate is therefore equal to

$$\overline{F} = A\bar{f} = \frac{\varepsilon_0}{2} A E^2 \hat{n}$$

As a result of this force, the plates of the parallel-plate capacitor move closer together by an infinitesimal distance ε. The work done by the electrostatic forces during this movement is equal to

$$W = \int \overline{F} \bullet d\bar{l} = \frac{\varepsilon_0}{2} A\, \varepsilon\, E^2$$

b) The total energy stored in the electric field is equal to

$$W = \frac{\varepsilon_0}{2} \int E^2 d\tau$$

In an ideal capacitor the electric field is constant between the plates and consequently we can easily evaluate the volume integral of E^2:

$$W = \frac{\varepsilon_0}{2} A\, d\, E^2$$

where d is the distance between the plates. If the distance between the plates is reduced, then the energy stored in the field will also be reduced. A reduction in d of ε will reduce the energy stored by an amount ΔW equal to

$$\Delta W = \frac{\varepsilon_0}{2} A\, d\, E^2 - \frac{\varepsilon_0}{2} A\, (d - \varepsilon)\, E^2 = \frac{\varepsilon_0}{2} A\, \varepsilon\, E^2$$

which is equal to the work done by the electrostatic forces on the capacitor plates (see part a).