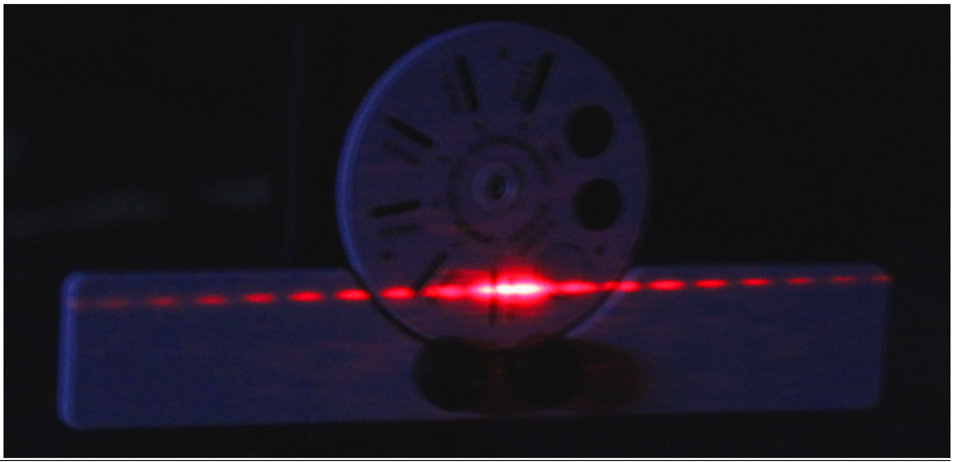
PHY143 LAB 2: DIFFRACTION



Introduction

In the geometric optics lab you observed light beams by tracing their paths as if they were a stream of individual particles traveling in straight lines. However, light also exhibits many wave behaviors that cannot be explained by the particle model. One such behavior is diffraction, which occurs when a wave meets an obstacle and scatters around it. As the wave spreads out after the obstacle different parts of the wave interfere with each other. The resulting pattern of maxima and minima form an interference pattern that is characteristic of both the obstacle and the wavelength of the wave being scattered.

Together with experiments such as the photo electric effect, diffraction experiments helped to establish the wave-particle duality of light. Similarly, diffraction experiments (performed by Clinton Davisson and G.P. Thompson) provided evidence for the existence of deBroglie's matter waves.

In this lab you will use a laser and a series of slits to generate interference patterns. First, you will use a single slit of a known width to determine the wavelength of the laser. You will then use this wavelength to measure the distance between two slits by recording their interference pattern.

THEORY

We can use the wave properties of light to make excellent models for the interference pattern of light. All these models employ approximations that greatly simplify the formulas. Since you will be working with slit separations on the order of 0.1 mm and measuring intensity patterns on the order of 1 cm with a screen distance on the order of 1 m, we will assume that
$d \ll x \ll L$.

Double Slit Maxima

First, we will look at a prediction of the maxima of a double slit diffraction pattern.

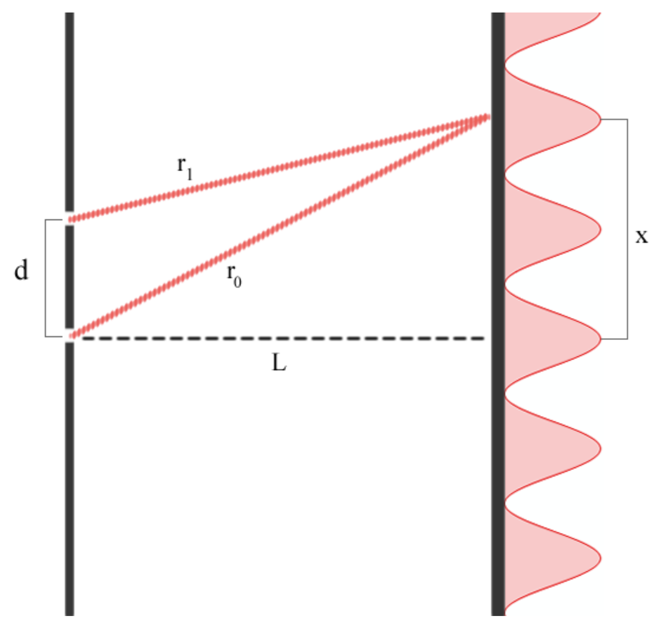


Figure 1

Here we have two slits separated by a distance d, both a distance L away from a screen. For some point a distance *x* along the screen, the distance to each slit is r_0 and r_1 respectively. Waves travel along r_0 and r_1 from the slits to meet and interfere at *x.* To calculate at which *x* the waves will interfere constructively to attain a maximum, we need to know where the waves are in phase when they reach the screen. Since the phases vary predictably along the paths of the waves, we need only calculate the path difference of the waves to determine where the constructively interfere. By the Pythagorean Theorem, we have

$$r_0^2 = L^2 + x^2$$

and

$$r_1^2 = L^2 + (x-d)^2 = L^2 + x^2 - 2xd + d^2 = r_0^2 - 2xd + d^2$$

.

Simplifying for r_1, we have

$$r_0 = \sqrt{L^2 + x^2} = L\sqrt{1 + \frac{x^2}{L^2}}$$

and

$$r_1 = \sqrt{r_0^2 - 2xd + d^2} = r_0\sqrt{1 + \frac{d^2 - 2xd}{r_0^2}}$$

Let

$$\alpha = \frac{d^2 - 2xd}{r_0^2}$$

Then

$$r_1 = r_0\sqrt{1 + \alpha}$$

Given our assumptions ($d \ll x \ll L$) we can see that α will be small. You should verify this, using typical values for the experiment, e.g. d = 0.01 mm, x = 10 cm and L = 1 m.

We want to approximate r_1 for small values of α, so we expand r_1 in a Taylor series about 0:

$$r_1 = L\left[1 + \frac{1}{2}\alpha - \frac{1}{2!}\left(\frac{1}{4}\right)\alpha^2 \dots\right]$$

With our assumptions, you can verify that

$$\frac{1}{2}\alpha \gg \frac{1}{8}\alpha^2$$

Thus we can ignore the α^2 term and those of higher degree, approximating

$$r_1 \cong r_0\left(1 + \frac{1}{2}\alpha\right)$$

We can then find the path length difference in terms of these expansions:

$$|r_0 - r_1| = r_0 - r_0\left(1 + \frac{1}{2}\alpha\right) = -\frac{1}{2}r_0\alpha = \frac{2xd - d^2}{2r_0} = \frac{2xd - d^2}{2L\sqrt{1 + \frac{x^2}{L^2}}}$$

Then, since $x \ll L$,

$$\frac{x^2}{L^2} \ll 1$$

and $2L\sqrt{1+\frac{x^2}{L^2}}$ reduces to $2L$. We also have that $2xd \gg d^2$, so the difference in path reduces to

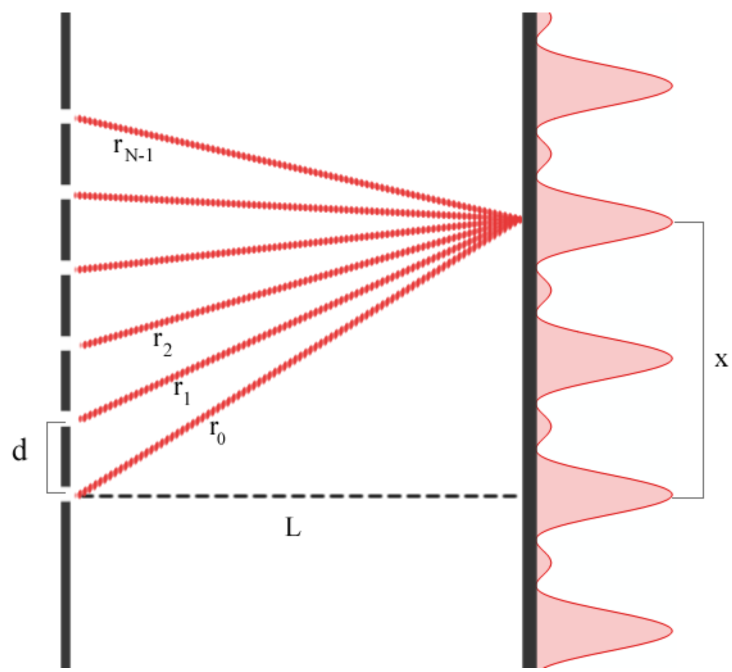
$$\frac{xd}{L}$$

Now that we've gotten by the tricky bit, we can finally determine where the maxima occur. Two waves with wavelength λ will constructively interfere when offset by distances of 0, λ, 2λ... or mλ for any integer m. Thus, for our double slit setup, we have

$$\frac{xd}{L} = m\lambda$$

which implies that maxima occur for every $x = \frac{m\lambda L}{d}$.

Multiple Slit Interference



Now we look at an approximation to the interference pattern of multiple evenly spaced slits. We extend our previous picture to *N* slits. We will be making similar distance approximations, but now we will explicitly find intensity as a function of *x* instead of simply finding the locations of the maxima. To do this, we employ the Huygens-Fresnel principle, which is the wave manifestation of Fermat's principle of least time (and the path integral approach to quantum mechanics). This principle treats each point of a wave as a source. For our purposes, we will continue to restrict our attention to two dimensions, as depicted in the figure. We assume (not quite so realistically) that each slit is infinitely thin; a point. By the Huygens-Fresnel principle, even though the actual source is behind the barrier with the slits, we can treat each point as a wave source. To obtain the interference at a point *x* on the screen, we sum up the contributions from each slit. Then we take the absolute square of this result (which is always a real number) to obtain the intensity.

First we look at the explicit expression for a point source. We will model each wave with a complex exponential that falls inversely with the distance from the source, which satisfies the wave equation. This is not as bad as it may sound, so suspend your potential befuddlement for the time being. You are going to have to learn this at some point anyway, as physicists frequently use this technique. Each wave as a function of distance from the source and time is given by

$$\phi(r,t) = \frac{Ae^{i(kr-\omega t)}}{r}$$

where *r* is the distance from the source, *t* is time, *A* is the amplitude, $k = \frac{2\pi}{\lambda}$, and ω is the angular frequency. It's fairly easy to gain some intuition about this equation. By Euler's equation, the complex exponential gives the phase of the wave as a function of its distance from the source and time, as you would expect from familiar waves like sine. Also note that if we take the absolute square of this equation, we get r^2 in the denominator, which we expect for the inverse square drop-off of intensity of light.

Now let's turn to the approximations, which proceed similarly to the double slit approximation. Since it is convenient to simply extend the double slit model, we will label our slits with integers $\mathrm{n} = 0, \dots, \mathrm{N}-1$, reflecting the first slit's position at the origin. The distance from the nth slit to a point x on the screen is given by

$$r_n^2 = L^2 + (x - nd)^2 = r_0^2 - 2xnd + n^2d^2$$

$$\Rightarrow r_n = r_0\sqrt{1 + \frac{n^2d^2 - 2xnd}{r_0^2}}$$

In spirit of the double slit approximation, let

$$\alpha_n = \frac{n^2d^2 - 2xnd}{r_0^2}$$

yielding the familiar looking expression

$$r_n = r_0\sqrt{1 + \alpha_n}$$

We can approximate using similar methods as we did previously, yielding

$$r_n \cong r_0\left(1+\frac{1}{2}\alpha_n\right)$$

You should verify that α_n and α_n^2 are appropriately small for the approximation to be valid, and consider what values of *n* are reasonable for this approximation.

Now, back to our waves, we can sum up the contributions from all slits at point x:

$$\psi(x,t)=\sum_{n=0}^{N-1}\phi_n(r_n,t)=\sum_{n=0}^{N-1}\frac{Ae^{i(kr_n-\omega t)}}{r_n}\cong\sum_{n=0}^{N-1}\frac{Ae^{i\left(kr_0\left(1+\frac{1}{2}\alpha_n\right)-\omega t\right)}}{r_0\left(1+\frac{1}{2}\alpha_n\right)}$$

Factoring out terms independent of our summing variable *n*, we obtain

$$\psi(x,t)=\frac{Ae^{i(kr_0-\omega t)}}{r_0}\sum_{n=0}^{N-1}\frac{e^{ikr_0\left(\frac{1}{2}\alpha_n\right)}}{1+\frac{1}{2}\alpha_n}$$

Now we apply our simplifications. Let's first look at the denominator. We have $d \ll x \ll L$ and *n* appropriately small, which implies that $1 \gg \frac{1}{2}\alpha_n$. This means that this term can be neglected, and the denominator can be approximated as 1. Looking at the exponential, substituting for α_n, we have

$$e^{\frac{ik(n^2d^2-2xnd)}{2r_0}}$$

To simplify this, we would really like $n^2d^2 \ll 2xnd \;\Rightarrow nd \ll 2x$. You can check that this is indeed a good approximation for reasonable values of *n*. Finally, we apply the same approximation $r_0 \cong L$ as we did previously.

With these approximations, the total wave function at a point *x* simplifies to

$$\psi(x,t)\cong\frac{Ae^{i(kr_0-\omega t)}}{L}\sum_{n=0}^{N-1}e^{-ik\frac{nxd}{L}}$$

This has the form of a geometric series. Letting $\beta = k\frac{xd}{L}$, we have

$$\sum_{n=0}^{N-1}(e^{-i\beta})^n=\frac{e^{-iN\beta}-1}{e^{-i\beta}-1}$$

Putting this in the wave function and taking its absolute square, we obtain an expression for the intensity at point *x* for *N* slits:

$$I_N(x) = |\psi(x,t)|^2 = \left|\frac{Ae^{i(kr_0-\omega t)}}{L}\right|^2 \left|\frac{e^{-iN\beta}-1}{e^{-i\beta}-1}\right|^2$$

Using Euler's equation and a few trig identities, (which you can verify) we get

$$I_N(x) = \frac{A^2}{L^2}\left(\frac{\sin\frac{N\beta}{2}}{\sin\frac{\beta}{2}}\right)^2$$

Let $\frac{A^2}{L^2} = I_1$ and substituting in for β and k, we finally obtain a useful formula

$$I_N(x) = I_1\left(\frac{\sin\frac{N\pi x d}{\lambda L}}{\sin\frac{\pi x d}{\lambda L}}\right)^2$$

Note that the function is undefined for $x = 0$, but we find that the function approaches $N^2 I_1$ as x approaches 0, so we simply define $I_N(0) = N^2 I_1$, making the function continuous. Note that for $N = 1$, we have $I_1(0) = I_1$, so I_1 is the central peak intensity of a single infinitely thin slit.

Single Slit Diffraction

Now let us consider a single slit with a *finite* width a. We take the same standpoint with the Huygens-Fresnel principle and treat each point in the slit as a wave source. This is essentially the same phenomenon as the multiple slit setup, but here we work with an infinite number of infinitely small slits placed within a width a.* Making this a little more concrete, let $a = Nd$ for N slits separated by d, and let $s = nd$ for some slit $n = 0, \ldots, N-1$. Then we can substitute s in place of nd in r_n and α_n to yield

$$r_s = r_0\sqrt{1+\alpha_s}$$

And

$$\alpha_s = \frac{(s^2 - 2xs)}{r_0^2}$$

Then, since $0 < s < a$, we can approximate

$$r_s = r_0\left(1 + \frac{1}{2}\alpha_s\right)$$

* This idea leads to the sum over all paths approach to quantum mechanics. For a good qualitative description of this concept, pick up the book *QED: The Strange Theory of Light and Matter* by Richard Feynman or look up his video lectures.

We can proceed as we did with the multiple slit case, but since we have to add up waves from an infinite of slits packed in the interval from 0 to a, we take the sum over $N-1$ to an integral along s by putting the sum into the form of a Riemann sum. Since we will be ultimately summing an infinite number of waves, we make each wave amplitude infinitesimal by replacing A with $\frac{A}{N}$.

$$\psi(x,t) = \sum_{n=0}^{N-1} \phi_n(r_n,t) = \sum_{n=0}^{N-1} \frac{A}{N} \frac{e^{i(kr_n-\omega t)}}{r_n}$$

We then multiply by $\frac{d}{d}$ so that the d may serve as the "width" of each summand in the Riemann sum, and the $1/d$ combines with N to yield a.

$$\frac{A}{Nd}\sum_{n=0}^{N-1} \frac{e^{i(kr_n-\omega t)}}{r_n}(d) = \frac{A}{a}\sum_{n=0}^{N-1} \frac{e^{i(kr_n-\omega t)}}{r_n}(d)$$

Substituting s for nd and taking $n \to \infty$ and $d \to 0$,

$$\frac{A}{a}\sum_{n=0}^{N-1} \frac{e^{i(kr_n-\omega t)}}{r_n}(d) \to \frac{A}{a}\int_0^a \frac{e^{i(kr_s-\omega t)}}{r_s}\mathrm{d}s$$

Applying the same rules of linearity to the integral as we did for the sum yields

$$\psi(x,t) = \frac{Ae^{i(kr_0-\omega t)}}{ar_0}\int_0^a \frac{e^{ikr_0\left(\frac{1}{2}\alpha_s\right)}}{1+\frac{1}{2}\alpha_s}\mathrm{d}s$$

Here we can make similar approximations as we did for the multiple slit case. You can verify that $1 \ll \frac{1}{2}\alpha_s$, $\alpha = \frac{s^2-2xs}{r_0^2} \cong -\frac{2xs}{r_0^2}$, and $r_0 \cong L$ for appropriate values of s. This leaves us with

$$\psi(x,t) \cong \frac{Ae^{i(kr_0-\omega t)}}{aL}\int_0^a e^{-ik\frac{xs}{L}}\,\mathrm{d}s = \frac{Ae^{i(kr_0-\omega t)}}{ikax}\left(1-e^{-\frac{ikax}{L}}\right)$$

We can finally take the absolute square, giving us

$$I(x) = |\psi(x,t)|^2 = \left|\frac{Ae^{i(kr_0-\omega t)}}{ikax}\right|^2\left|1-e^{-\frac{ikax}{L}}\right|^2$$

Applying identities as we did before and simplifying yields

$$I(x) = \frac{A^2}{(kax)^2}4\sin^2\frac{kax}{2L} = \frac{A^2}{L^2}\frac{\sin^2\frac{kax}{2L}}{\left(\frac{kax}{2L}\right)^2}$$

Letting $I_0 = \frac{A^2}{L^2}$ and substituting for k, we finally get

$$I(x) = I_0 \left(\frac{\sin \frac{\pi a x}{\lambda L}}{\frac{\pi a x}{\lambda L}} \right)^2$$

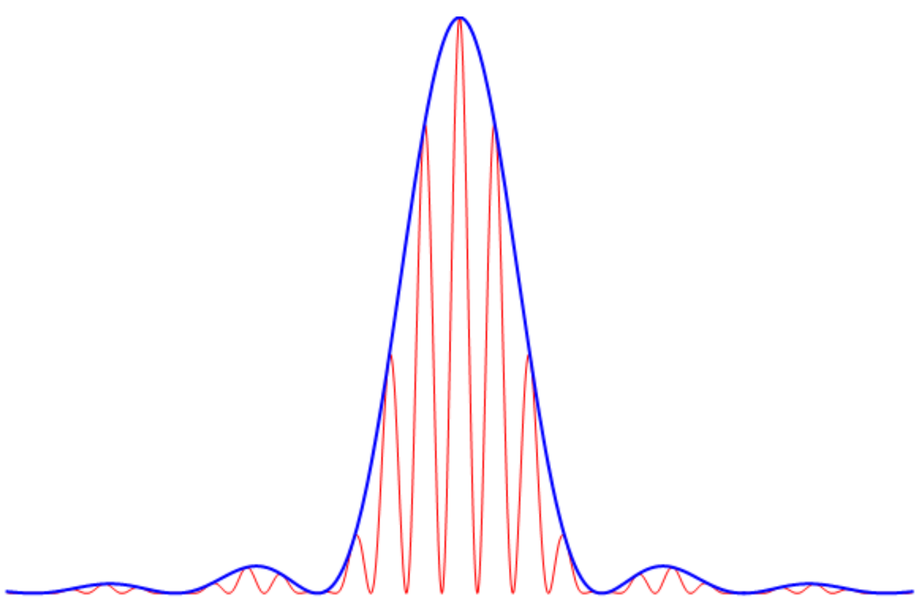
Where I_0 is the central intensity of the diffraction pattern.

Multiple Slit Diffraction

Consider now that we have *N* slits of finite width *a* spaced distance *d* apart. Although we will not explicitly derive the expression for the diffraction pattern, we will compare it to our previous results. The intensity as a function of position is given by:

$$I_N(x) = I_1 \left(\frac{\sin \frac{\pi a x}{\lambda L}}{\frac{\pi a x}{\lambda L}} \right)^2 \left(\frac{\sin \frac{N \pi x d}{\lambda L}}{\sin \frac{\pi x d}{\lambda L}} \right)^2$$

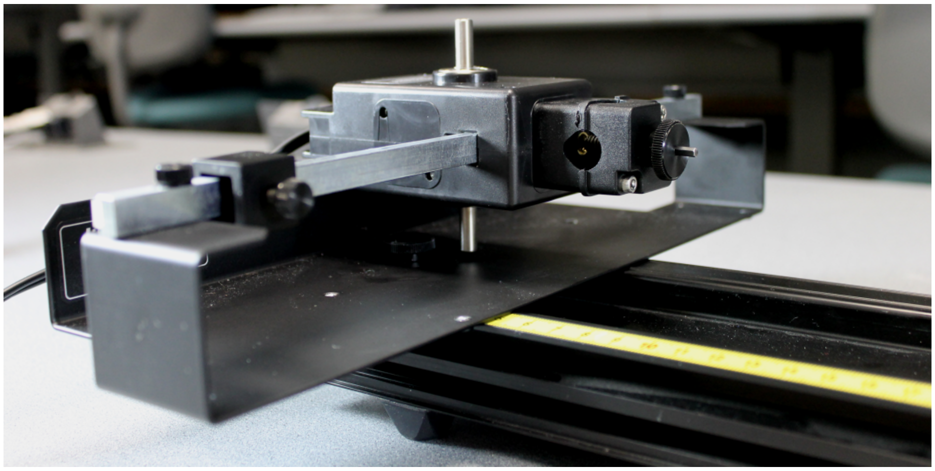
where I_1 is again the central peak intensity of a single slit pattern. Note that this is the product of the intensity pattern for multiple infinitely thin slits and a single slit of finite width. We can view this as the multiple slit interference pattern *modulated* by the single slit diffraction pattern. The values of the single slit diffraction pattern (times N^2) for some a bound the function above for any choice of *x*, *N* and *d*. We say that the single slit pattern is an *envelope* for I_N. Since the envelope is independent of the distance between the slits, it can be used to determine a.



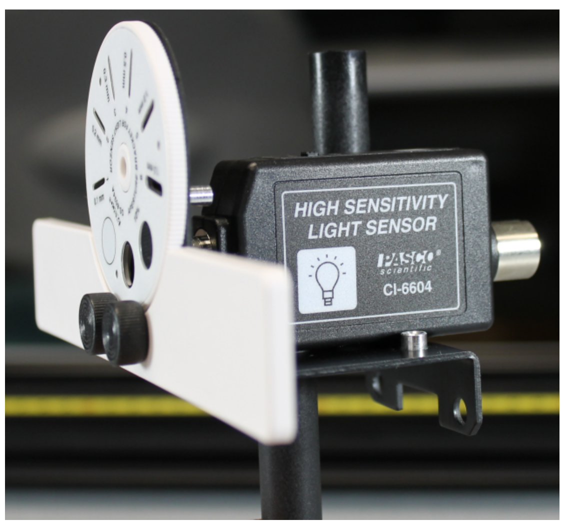
EXPERIMENT

Experiment Setup

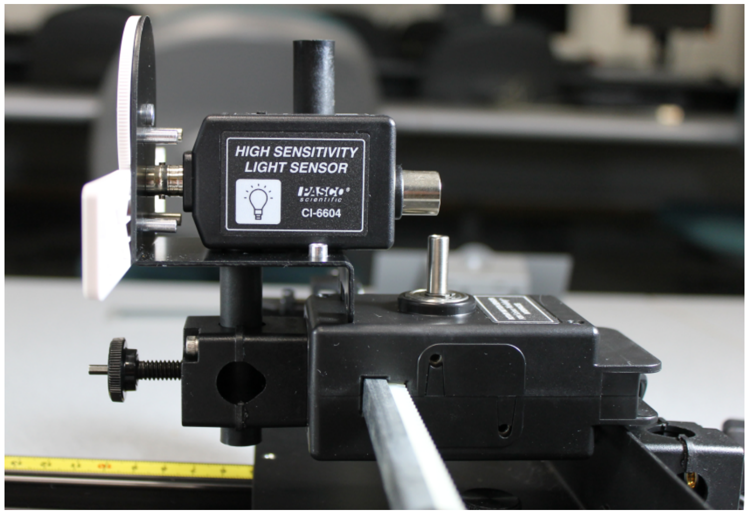
1) Place the Linear Translator at one end of the optics bench.



2) Mount the High Sensitivity Light Sensor on the Aperture Bracket using the black rod (if you have two either is fine). Which aperture slit should you chose for the highest resolution? What about to collect the most light? What gain should we use for the experiment? Why?

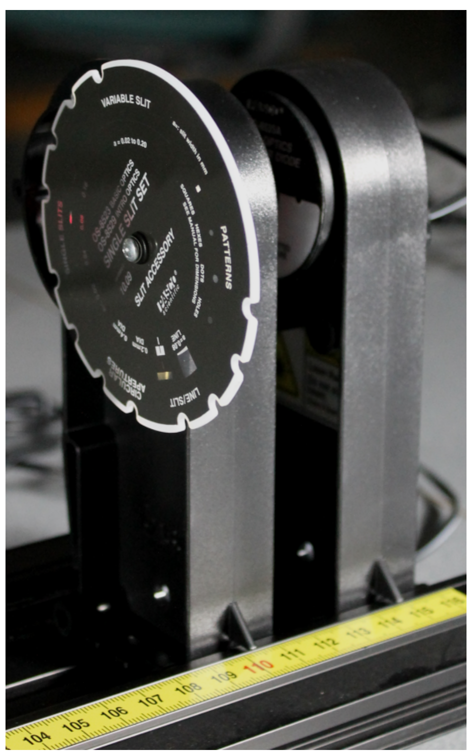


3) Place the black rod in the Rotary Motion Sensor clamp found on the front of the Rotary Motion Sensor. Make sure that the High Sensitivity Light Sensor is parallel to the track.

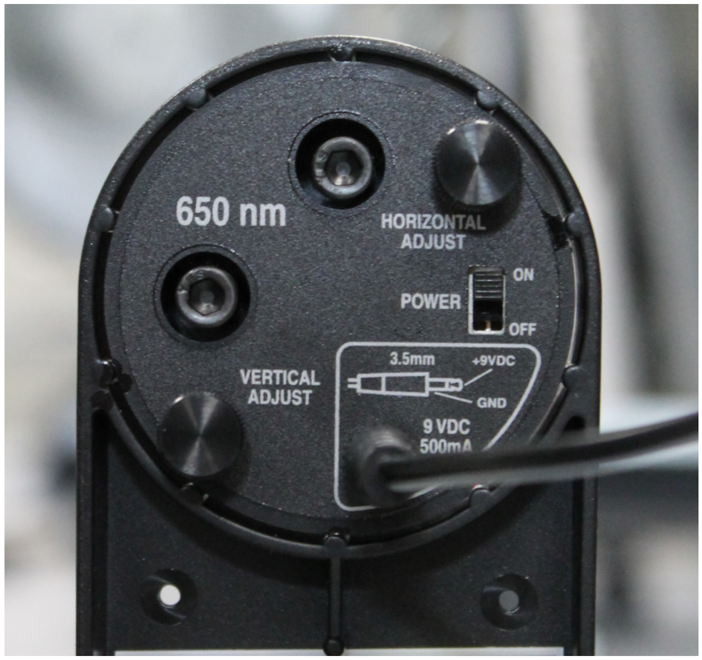


4) Place the laser diode on the other end of the optics bench. Plug in the power cable for the laser and turn the laser on. IMPORTANT: do not shine the laser into anyone's eye; do not look into the laser.

5) Place the slit holder on the bench in front of the laser. Should you maximize or minimize the distance between the laser and the sensor? What about the distance between the laser and the slits? Be sure to record the distance between the Slit Wheel and the face of the Aperture Bracket.



5) Use the horizontal and vertical adjustment knobs on the back of the laser to aim the laser at the slit until it passes through. You should now see a diffraction pattern on the front plate of the Aperture Bracket. Rotate the disk of slits in its holder until the pattern is horizontal.



Computer setup

1. Turn on the ScienceWorkshop 750 Interface.

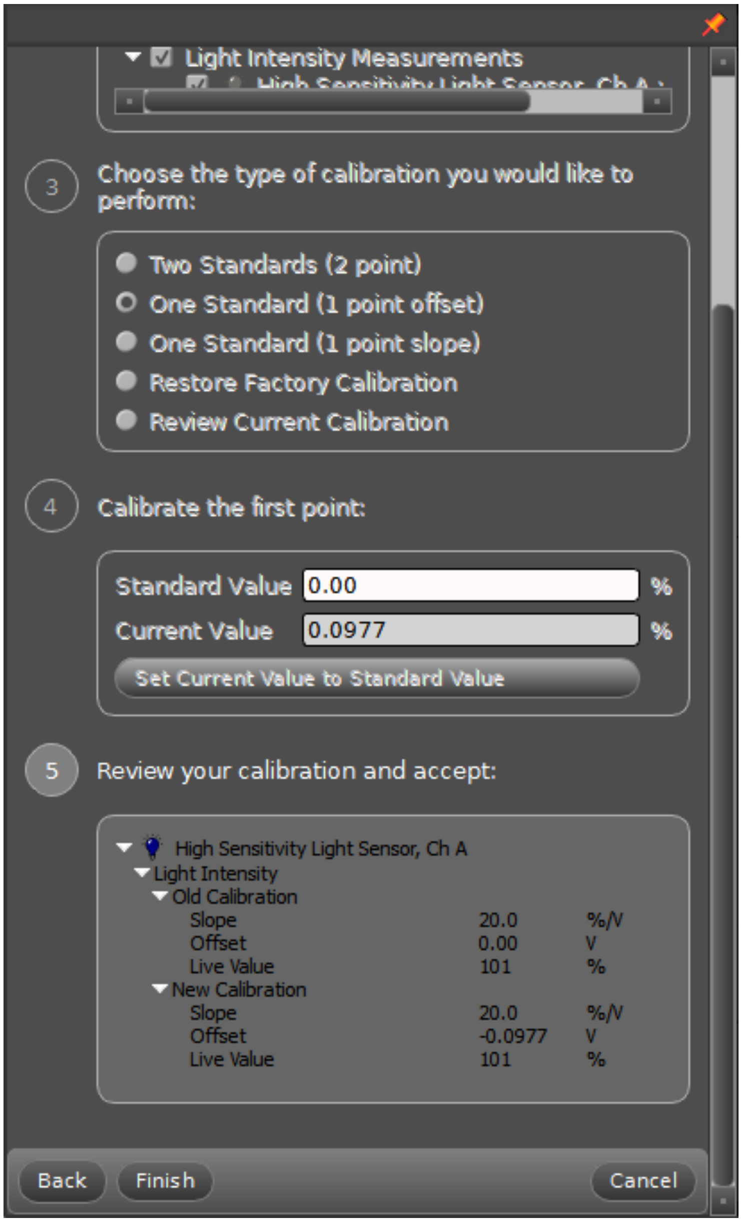
2. Connect the cable between the light sensor and Analog Channel A. Plug the rotary motion sensor into Digital Channels 1 and 2.

3. Launch Capstone and load the diffraction lab setup file from the PHY143 website.

4. Press record and move the rotary motion sensor. If the points are plotted negatively, press stop and reverse the rotary motion sensor's inputs.

5. Calibrate the High Sensitivity Light Sensor. Click Calibration (the green circle on the left side of the screen). From the menu, select Light Intensity and click next. Select the dot that says One Standard (1

point offset) and click next. Cover the Light Sensor then click Set Current Value to Standard Value. Click Finish. Click Calibration again to close the calibration menu.



- What are the errors on your measurements?
- How many data points do you need to take for each experiment?
- How many data points do you need to take for each experiment?
- How does the speed with which you move the sensor affect your results?
- How would a poorly calibrated light sensor change your data?

Experiments

For both experiments, perform the calculations using the peak finder in IGOR pro or in Capstone to find the location of each peak. Then fit your data in IGOR pro and compare the best fit values to your calculated results.

1) Select a single slit. Using the slit width marked on the slit wheel, calculate the wavelength of the laser.

2) Select a double slit. Using the slit width marked on the slit wheel and the wavelength calculated in part (1), determine the spacing between the two slits.
3) Record data from one set of more than two slits. Fit the data in Igor Pro to determine the width of the slits.

Some useful information

- The Rotary motion sensor and linear translator are accurate up to 0.055 mm[†]
- The labeled slit width and slit spacing values on the slit accessory wheel are accurate to ±0.25 mm[‡]
- The Light Sensor has a resolution of approximately 10 micro lux[§]

[†] http://www.pasco.com/prodCatalog/OS/OS-8535_linear-translator-basic-optics/index.cfm
[‡] http://www.pasco.com/prodCatalog/OS/OS-8523_slit-accessories-basic-optics/index.cfm#specificationsTab
[§] http://www.pasco.com/file_downloads/product_manuals/Light-Sensor-Manual-CI-6504A.pdf